HIGH SCHOOL MATHEMATICS

FIRST COURSE

TEACHERS' EDITION

UNIT ONE 1957-58

DIRECTED NUMBERS

UNIVERSITY OF ILLINOIS
COMMITTEE ON
SCHOOL MATHEMATICS

Math

Unit 1

DIRECTED NUMBERS

1.01 Arithmetic by mail.--What kind of an error is Paul making when he adds 5 and 7 and gets 57 for an answer? Or when he says that 9 goes into 99 twice?

1.02 Numbers and their names.--It is important to recognize when a book is talking about symbols and when it is talking about the things for which the symbols stand.

1.03 Distances and directions.--The number +2 corresponds to a trip made in one direction and the number -2 corresponds to a trip in the opposite direction.

1.04 Adding directed numbers.--To obtain the result of adding two directed numbers, combine two trips corresponding to these numbers.

1.05 Using directed numbers.--Directed numbers can be used whenever measurable changes take place in one of two opposite directions.

1.06 Multiplying directed numbers.--One can learn to multiply directed numbers by considering a pump which pumps in or out of a tank in which water rises and falls and then considering a motion picture of this which runs forwards or backwards.

1.07 A book sale.--By using a table, one can add or multiply certain numbers in a surprising way.

1.08 Positive numbers and the numbers of arithmetic.--In adding, multiplying, subtracting, and dividing, the positive numbers "act like" the numbers of arithmetic.

1.09 Using symbols of grouping.--Parentheses, brackets, and braces are sometimes necessary to tell what number is intended.

1.10 Directed numbers and the principles of arithmetic.--There are several familiar principles (associativity, commutativity, distributivity, etc.) of arithmetic. These principles also apply to directed numbers.

1.11 Subtracting directed numbers.--Knowing that subtraction is the inverse of addition one can subtract by converting subtraction problems to addition problems.

1.12 Dividing directed numbers.--Division is the inverse of multiplication. Therefore, one can convert division problems into multiplication problems.

1.13 Comparing numbers.--Subtraction gives us a method for deciding which of two directed numbers is larger.

1.14 The number line.--The directed numbers may correspond to the points on a line.

UICSM
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TEACHERS COMMENTARY

Introduction

The text materials for the UICSM program are produced by high school teachers, educators, mathematicians, and scientists, all on the staff of the University of Illinois. Since 1951, we have been debating the issues:

> What mathematical ideas should be taught in high school?
> What are the most effective ways to teach these ideas?

We have always felt that the teachers participating in the Project ought to "take part" in these discussions. The TEACHERS COMMENTARY is an attempt to let the participating teacher in on some of them.

The COMMENTARY also brings you the experiences and suggestions of those of you who have worked with us last year and in earlier years. We have used the daily reports you sent us not only in revising the student's materials but in adding good teaching suggestions to the previous edition of the COMMENTARY.

We continue to welcome all kinds of suggestions from you--your ideas, student reactions, samples of their work, complaints and praise from parents, and even reports of the times when you felt that "the Illinois people were just crazy".

Our operating principles

For the success of the program it is important that you adopt a spirit in teaching similar to the spirit we try to adopt in our writing. In our work we recognize several basic principles which are easy to state but not always so easy to carry out.

(I) The learning of mathematics should be a delightful experience for youngsters.

In practical terms, this means that the development of a mathematical idea will have to be couched in situations which are inherently interesting to young people. [Some people have interpreted this principle to mean that students are interested only in "real life" applications. Yet, many of the things in which your students are interested are hardly in accord with the adult's view of "real life".] One of our standard devices when approaching a new idea is to create a fanciful situation which embodies or illustrates the idea. A student will know that these situations would never occur. Nonetheless, he can easily imagine them.

An important ingredient in effective learning is the creative activity of the learner. Our materials are full of opportunities for creative activity of the intellectual variety.

(II) Mathematics can be interesting without being watered-down.

In fact, watered-down mathematics cannot be interesting. All of you know that a watered-down course is a time-waster for both the talented student and the less able student. We have tried to provide in our materials an intellectual challenge for a wide range of ability. The COMMENTARY will include suggestions for making a fuller use of this flexibility.

(III) The teacher is an extremely important element in the classroom.

Our text materials are not of the self-teaching variety. Despite the overcrowded classrooms with which most of you have to work, we do not believe that most students can learn mathematics independently of the teacher and class. For example, you will often find a series of exercises which are designed to take the student to successively higher levels of abstraction. In the COMMENTARY, we shall point out the various levels of abstraction we are hoping for, so that you can better help the student as he works through the exercises. This means, of

course, that many of the exercises should be worked in class under your supervision.

You will find it necessary to "fill in the gaps" in certain parts of the materials. By this we mean that we have not tried to explain each and every detail of a process or proof or other development. An accumulation of such details tends to make textbooks too ponderous. We expect the teacher to supply these details to his class whenever he feels the class can tolerate them. Of course, we hope that the COMMENTARY plus the student text will give the teacher all of the necessary information to do this. But the teacher is the judge of how deep and how far to go with a given class.

Style and format of the COMMENTARY

A glance at the green pages which occur in this unit will show you how the COMMENTARY is arranged physically. We think this arrangement will be convenient for you. When you receive a teachers' edition of a new unit, you may find it helpful to read all of the white pages fairly rapidly. Then, read the unit again, this time including the green pages. Naturally, you will want to read smaller sections of the green and white pages each day as you prepare for your classes.

We plan to use an informal style in writing the COMMENTARY. We think you will find the COMMENTARY more useful if you can pretend that you are conversing (one way, alas!) with one of us at the Project center. You will have a chance to discuss all questions with us during our visits to your school and during your conferences with us in Urbana.

Acknowledgment

The UICSM Project Staff takes this opportunity to thank teachers of FIRST COURSE for their detailed written reports, for the suggestions and criticisms which they gave us in conversation and during training conferences, and for the good ideas that came to us as we worked with the students in their classes.

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Mr. Paul Dietz and Mr. Howard Marston, The Principia Upper School, St. Louis, Missouri

Mr. George Baptist and Mr. John Friedlein, St. Charles High School, St. Charles, Illinois

We thank the many members of the profession who took time to send us their comments concerning FIRST COURSE. We hope that the teachers, curriculum study groups, mathematicians, scientists, parents, and college students who have an opportunity to read this Teachers Edition of FIRST COURSE will send us their comments so that we can consider them for future editions.

We also thank the administrative officials of the University of Illinois and the Carnegie Corporation of New York for their generous support of this Project.

UICSM Project Staff
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Pages 1-1 through 1-4 are intended as a humorous introduction to the course. They embody an extremely important idea which will come up time and again in teaching UICSM materials. The idea is developed further in pages 1-5 through 1-9. But it may be possible to get at the heart of the matter through the humorous interchange between Paul and his "tutor", Ed.

* * *

The Project Staff has presented its point of view concerning the importance of distinguishing between symbols and their referents in the article "Words, 'Words', "Words" " which appeared in the March 1957 issue of The Mathematics Teacher. You can obtain a reprint of this article by writing to us.

1.01 Arithmetic by mail.--Ed Brown had a young friend, Paul Moore, who lived in Alaska. Paul and Ed corresponded quite frequently. Ed liked to receive letters from Paul because he wrote about such interesting things as hunting and fishing and prospecting for gold. Paul enjoyed hearing about the things Ed did in the United States, especially about school, for he had had very little opportunity to attend school in Alaska. One day he wrote to ask if Ed would mind teaching him some arithmetic. Ed agreed but decided he needed to know how much Paul already knew. So, in his next letter to Paul he included a simple test and asked him to write in the answers and to return the test to him. Paul sent the test back immediately; he said it was very easy and asked Ed to send some harder questions next time.

Turn the page to see what Paul's test looked like when he returned it:

We suggest that you select a student to read aloud the interchange of correspondence between Paul and Ed. It is best to select a student who has some flair for dramatics because the humorous flavor of the letter must be brought out in the reading. [You may find it necessary to read the letters aloud yourself.] Undoubtedly, you will find students who disagree with Paul's justifications and with Ed's justifications. This disagreement should be encouraged even if it comes close to hair-splitting.

* * *

Your students ought to see that Paul is perfectly consistent in the way he has responded to Ed's questions. Let the students tell why Paul answered as he did for each item. Be graphic in describing why '3' goes into '8' twice. You might want to bring a paper '8' into class and actually cut it into 2 '3's!! Let the students devise similar questions and answer them as Paul would, even questions outside of mathematics such as:

Is MARY bigger than mary?

* * *

Ask your students to try their hands at writing to Paul before they read Ed's letter. Then let them read Ed's letter.

1.	Take away 2 from 21.	....1....
2.	What is half of 3?	....
3.	Add 5 to 7.	...57....
4.	Does $2 \times 4\frac{1}{2}$ equal 9?	..no....
5.	Which is larger, .000065 or .25?	.000065
6.	How many times does 3 go into 8?	twice..
7.	How many times does 9 go into 99?	twice..
8.	Which is larger, 3 or 23?	...23....
9.	Give a number smaller than 4.	....4.....
10.	Give a number larger than 4.	....4.....

Ed was flabbergasted when he looked at Paul's answers. Was this a joke? But Paul had seemed so serious about wanting to learn arithmetic. Ed decided that Paul needed a lot of help. He would start by telling Paul about the errors he had made.

Dear Paul,

You sure have some funny ideas about numbers. But it won't take long to straighten you out (I hope).

I've enclosed your test in this letter so that you could follow my explanation. Look at the first question. I can see how you thought that the answer was 1 because if you <u>really</u> do take 2 away from 21 you are left with 1. But when you take 2 away from 21 you should get 19. See? Take the second question. Your answer isn't even a number, it's only half of a number. What you wrote is half of 3 but

When Ed says that Paul is right for item 6, Ed is thinking of division without remainder, of course. However, Paul actually cut the '8' into two pieces.

* * *

On page 1-4 Paul explains more of his techniques.

half of 3 isn't what you wrote, it's $1\frac{1}{2}$. In the third question you put 5 and 7 together but adding numbers isn't putting them together. It's, er, well ..., it's ...er. Let's come back to that one in your next lesson. In question 4, you ought to know that $2 \times 4\frac{1}{2}$ is the same as 9. Now in number 5 it's true that .000065 is bigger than .25, but really .25 is larger than .000065. You see, .25 is a bigger number even though it looks smaller. Now, number 6 you did correctly. Also number 8. So I won't have to explain those. I guess you just made a careless error in dividing number 7 because there are 11 9's in 99. You made the same kind of mistake in question 9 as you did in question 10. The number you wrote for question 9 sure is small but it's just as big as 4. The number you wrote for question 10 is much bigger than 4 but it's still the same size.

I hope that my explanations are helpful to you. Let me know when you want your next lesson.

Your pal,
Ed

Paul was a little disappointed with Ed's letter. He wrote the following in return:

Dear Ed,

I sure appreciate what you are trying to do for me but I don't think that _you_ can help me at all. Are you sure _you_ understand this stuff?

Sure I got numbers 6 and 8 right. I did them just as I did the others. Anyone can see that 3 goes into 8 twice, and pretty neatly, too. You put 3 into 8 the regular way and then you turn another 3 around and put it in on the other side of the 8. In question 8, you can just see that

By careful questioning try to elicit from the students the notion that what Ed needs is a writing system which will permit him to indicate when he is talking about things and when he is talking about the names of these things. You and the class may want to consider more examples like Part I on page 1-75.

* * *

Further amusing statements overheard in the printshop:

"Find a 2 bigger than this 8."

"Bring me 0 type."

"0.00001 is certainly bigger than 14."

23 is larger than 3 because 23 already has a 3 in it and a 2 added on.

I really laughed at what you said about question 7. If there's one thing I know it's how to count, and 11 9's make 99999999999 and not 99. You said that .000065 is bigger than .25. I knew that because I even checked with a ruler. Then you cross yourself up and say that .000065 is really smaller. And in question 4 you don't need a ruler to tell that $2 \times 4\frac{1}{2}$ is not the same as 9.

There's no use in trying to learn arithmetic, I guess. I think I'll stick to hunting.

Your friend,
Paul

EXERCISES

A. Do you think Ed did a good job explaining Paul's errors? What would you have written to Paul? What seemed to be Paul's chief difficulty?

B. In certain print shops the printing press is prepared for operation by picking individual type pieces out of boxes and placing the pieces into the bed of the press. Such a print shop has type of various styles and sizes. Here are two examples of remarks you might hear a printer make to his helper:

> Bill, this seven isn't big enough; get me a bigger seven.
>
> I asked you to bring me two threes and you brought me three twos. I don't care what you learned in school, three twos are not the same as two threes.

Make up more examples of confusing statements you might hear in this print shop.

When you [illegible] the [illegible] partly [illegible]

the girl Mary. [illegible]

[illegible]

[illegible] *The Negro* [illegible]

[illegible]

When you emphasize that the marks are not the numbers themselves, perceptive students may ask what the numbers are. They write the name 'Mary' and know what this name stands for because they know the girl Mary. On the other hand, when they write the name '2', they cannot see or touch the number 2. The question, 'What is the number 2?', is a very profound one, and is one which has come to be answered in a satisfactory way only recently. Tell students that, for the present, they should regard the number 2 as an abstraction, as something which "resides" in their minds. A good analogy is the distinction between the word 'justice' and the concept of justice. No one thinks that 'justice' is justice, yet the concept of justice is quite real to students. Justice is an abstract thing, and so is 2.

Read what is said about what numbers are in the book by H. A. Thurston, The Number System (London: Blackie, 1956).

* * *

Be sure the students see 'George Washington' in the upper right margin and 'Thomas Jefferson' on the next page.

George Washington

1.02 *Numbers and their names.*--It is easy to see what Paul had in mind when he took Ed's test. He was confusing numbers with the marks that appeared on the paper. When we write about numbers, we use words or marks on paper. Here are examples of these marks:

$$7 \qquad 6 - 2 \qquad 8\frac{1}{2} \qquad 4 \times 3 \qquad 0$$

Some of the marks which we write on paper are symbols for numbers. *The marks are not the numbers themselves.* Whenever you read a book that is trying to teach you how to do mathematics, many of the instructions you read will be instructions about what to do with the *symbols* in mathematics. In order to understand these instructions you must be able to tell when the book is talking about the symbols and when it is talking about the things for which the symbols stand.

Have you noticed that George Washington is on this page and that Thomas Jefferson is on the next page and that when the book is closed they are very close together? Did this sentence sound funny to you? It should have because George Washington and Thomas Jefferson have been dead for a long time, and they were never in a book, even when they were alive.

When you read the words:

George Washington

you think of the man. If a book intends that you think of the words themselves, it must tell you so by giving you a signal. The signal, for example, could be a loop drawn around the words, like this:

> Have you noticed that George Washington is on this page and that Thomas Jefferson is on the next page and that when the book is closed they are very close together?

When this use of words is intended, they may be printed in

We hope this discussion may lead to lunchroom play concerning whether you drink 'milk' or milk.

* * *

The students and you may want to agree upon a procedure for speaking in which a signal is given to serve the same purpose as single quotation marks. For example, waving the arms before and after speaking the word, or just saying, "You can't drink quote milk quote but you can drink milk."

* * *

Many kids enjoy the joke about the pencil that can write any color. Try it with a piece of white chalk by stating that you have a piece of chalk that can write any color:

red green blue

* * *

You can write names of names by extending the single-quotes device:

John is a boy.

'John' is a boy's name.

' ' John ' ' is a name for the name of a boy.

Read more about names in Alfred Tarski, Introduction to Logic (New York: Oxford University Press, 1954).

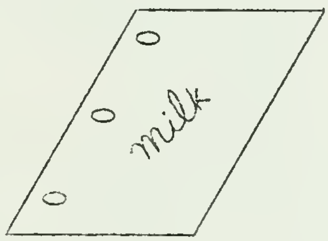
* * *

The boxed statement should be "old hat" for the kids by the time they reach this point.

italics, or they may be underlined, or they may be printed in bold-faced type. Can you think of other signals? In this book we shall put single quotation marks around the words when we want you to think of the words themselves rather than of the things for which the words stand. Often, instead of using single quotation marks, we may display the words on a separate line. When a new term is introduced, we shall underline it instead.

Read the following paragraph carefully and note the use of single quotation marks in talking about names:

> Some people put milk on their cereal. I have never seen anyone put 'milk' on his cereal. If you wanted to have 'milk' on your cereal, you could write 'milk' on a piece of paper and drop the paper into your cereal. Strangely enough, if you put 'milk' on a piece of paper, the piece of paper looks like this:



> If you put milk on a piece of paper, the paper becomes wet and gooey. Some people put sugar and milk on cereal. Webster prints 'sugar' and 'milk' on different pages. In Illinois quite a lot of milk is consumed. In 'Illinois' there are eight letters. People can use milk in a room without 'milk'. However, if there is a milk container in the room, it is likely that 'milk' is printed on the container. Thus, a milk container has milk in it and 'milk' on it.

To decide whether or not to put single quotation marks around a word, ask yourself:

Am I talking about the thing to which the word refers?
or
Am I talking about the word (or name) that I am writing?

In [illegible] and [illegible] are names for [illegible] the other hand, th[illegible]:

[illegible]

[illegible] and 7 [illegible] are names for [illegible].

There will [illegible] point out [illegible] when the [illegible] the same (or, names for [illegible]).

* * *

[illegible] the [illegible] one [illegible] I [illegible] did [illegible] and [illegible] Which [illegible] is [illegible] under [illegible] what the [illegible] number [illegible].

is true, because '3 + 1' and '9 - 5' are names for one number. On the other hand, the statement:

$$3 + 1 = 7 \times 5$$

is false, because '3 + 1' and '7 × 5' are names for two numbers.

There will be many times during the course when you will need to point out to the students that a statement containing '=' is true only when the expressions written on either side of '=' are names for the same number [or, names for the same thing].

* * *

Some of you may disagree with our use of the word 'numeral' as a name for a number. We have had difficulty finding an appropriate single term. The dictionary is unclear as to what a numeral is. We have decided that 'numeral' is an excellent generic term for the class of number names. Encourage the student to use 'numeral' when they refer to a name of a number.

Students in Mr. Fildes' class were quick to point out that the statement:

Ruth has four letters.

is ambiguous. It could mean that Ruth has received four letters through the courtesy of the United States Post Office or that Ruth has won four letters in athletics (or that the word 'Ruth' should have semi-quotation marks around it).

Thus, some students will say it makes sense and others will say it does not. In any case, a student should be able to defend his stand.

* * *

This display of symbols for 4 is the beginning of our campaign to get students to understand that a symbol such as '3 + 1' is <u>not</u> a signal to add 3 and 1 but is merely a name for 4. In fact, the symbol '3 + 1' is a name of the sum of 3 and 1. We think this will clarify eventually such questions as "you can't add 2 and $\sqrt{2}$ " and "you can't add $\frac{1}{3}$ and $\frac{1}{5}$".

Students seem to have a hard time understanding that '$2 + \sqrt{2}$' <u>is</u> a name for the sum of 2 and $\sqrt{2}$; we do not have a shorter name for this number. Similarly, '$\frac{1}{3} + \frac{1}{5}$' <u>is</u> a name for the sum of $\frac{1}{3}$ and $\frac{1}{5}$; a shorter name for this sum is '$\frac{8}{15}$'.

Here is another point which will come up again and again. When one writes two numerals for numbers, and puts '=' between the numerals, one has a statement which is either true or false. For example:

$$3 + 1 = 9 - 5$$

(continued on T. C. 7B)

In the first case you do not use quotation marks; in the second case you do.

Which of the following four sentences do you think make sense?

Ruth has blond hair.

'Ruth' has blond hair.

Ruth has four letters.

'Ruth' has four letters.

Be sure you understand that

Ruth is a girl

but

'Ruth' is her name.

Many mathematical symbols are names of numbers. So, when we talk about the *symbols themselves*, we shall use single quotation marks. For example, the symbol '4' is a name for the number 4. The symbol '$7\frac{1}{2}$' is a name for the number $7\frac{1}{2}$. Now, just as there may be many different names for the same thing, there may be many different names for a number. For example, the following symbols (except one symbol; find it!) are some of the names for the number 4:

IV $2 + 2$ four $7 - 3$ $8 - 4$ 4×0 $\frac{1842 - 1834}{2}$

4   $1 + 1 + 1 + 1$ $\frac{6 + 2}{2}$ 2×2 $3 + 1$ $3 \times 1\frac{1}{3}$ IIII

quatre 1×4 4×1 $628.424 \div 157.106$

$4 + 0$ $1 + 3$ 4 $72 \div 18$

We shall call a symbol or name for a number a *number symbol* or an *expression for a number* or a *numeral*. All (except one) of the symbols above are numerals for 4.

Some of the following statements are about the number 4; some are about names for 4:

One of our readers, Mr. Trenchard More, pointed out that after the student puts single quotation marks around 'Coal' in Exercise 5, the resulting sentence is false because 'Coal' occurs once and "coal" occurs once. Some of your brighter students may appreciate this point. It is a subtle one. If they understand this point, they should be able to see that the following sentence is true:

'California' does not have single quotation marks around it.

* * *

One student pointed out that the sentence in Exercise 6 could be punctuated as follows:

You might hurt me if you throw sticks at
me but not if you throw sticks at 'me'.

* * *

Another student pointed out that the sentence in Exercise 12 is correct as it stands if one thinks of the situation in which Mary's remains are located in a cemetery in Maryland. We shall not enter into a discussion of whether Mary's remains <u>are</u> Mary.

4 is an even number.

'4' is a numeral for 4.

'4' is not a number.

'2 + 2' is a name for 2 + 2.

'2 + 2' is a numeral.

2 + 2 is the sum of 2 and 2.

'IV' is not a number.

You can put '4' on paper.

You can't put 4 on paper.

You can't put 4 on paper anymore than you can put lions on paper. You can write 'lions' on paper but you usually put lions in a cage. (If you put a large sheet of paper on the floor of their cage, then it would be possible to put lions on paper!)

EXERCISES

Copy the following sentences and, whenever necessary, use single quotation marks around some of the words so that the sentences become reasonable statements.

1. Bill has a dog.
2. Bill found dog in a book.
3. In drawing class I have trouble with my pen when I make a 3.
4. Coal is mined in Pennsylvania.
5. Coal occurs twice on this page.
6. You might hurt me if you throw sticks at me but not if you throw sticks at me.
7. Fourteen is an even number.
8. Add 14 and 2.
9. Use the symbol 3 when you mean the number 3.
10. You can write the numeral 12 when you mean the number 5 + 7.
11. You can write 12 instead of writing 5 + 7.
12. Mary is a part of Maryland.
13. 2 is a part of 23.

(continued on next page)

In the first sentence of Section 1.03, we mention whole numbers and fractional numbers. Instead of this, we should refer to the numbers of arithmetic. These are the so-called unsigned real numbers. They include numbers like 7, $\sqrt{2}$, 96%, and π. We shall make frequent reference to the numbers of arithmetic in this course. [Later in the course, we use 'whole numbers' as a synonym for 'positive and negative integers and zero'.] We regard a fraction such as '$\frac{2}{3}$' as a symbol. A fraction is a collection of marks consisting of three parts: a numerator, a bar, and a denominator. Thus, the numerator and denominator of a fraction are symbols, not numbers. [The numerator of the fraction '$\frac{4}{5}$' is '4'; the numerator-number is 4.] A fractional number is named by a fraction. A fractional number may also be named by a percent or by a decimal fraction. Percents and decimal fractions are symbols, just as fractions are symbols. Thus, the fractional number $\frac{4}{5}$ has the names '$\frac{4}{5}$', '80%', and '0.8', among others.

Use the phrase 'numbers of arithmetic' when you mention the numbers the student learned about prior to this course.

the referents mentioned are two symbols, and the symbols <u>used</u> are names of these symbols.

* * *

In the box, we state that we sometimes underline a word instead of enclosing it in semiquotes. We also underline words for other reasons, such as to give emphasis.

Semiquotes are not used in displayed expressions when the displayed expression is preceded by a colon. [Some expressions are displayed because the display helps comprehension.]

<u>Examples</u>:

<u>No display</u>:

Most eighth graders know that '7×3' is a name for 21.

<u>Display without semiquotes</u>:

Most eighth graders know that:

$$7 \times 3$$

is a name for 21.

<u>Display for ease in reading</u>:

Most eighth graders know that

$$7 \times 3$$

is 21.

[Note that in the third example we are talking about (mentioning) a number, while in the first two we first mention a symbol and then a number.]

* * *

(continued on T. C. 9E)

Incidentally, a frequently used synonym for the term 'single quotation marks' is 'semiquotes'.

* * *

Reverend Bezuzska has suggested a useful way of thinking about the semiquotes device. When you read a sentence which contains a symbol in semiquotes, the semiquotes tell you to look at what is between them and think of that mark [or, perhaps, the class of all marks which are equiform with respect to the given mark]. Stop at the symbol, and don't go "behind" the symbol. When you come upon a symbol which is not enclosed in semiquotes, you look at the symbol and think about the thing (referent) for which the symbol stands.

In technical works on logic, the notion of distinguishing in written language between symbols and their referents is often called <u>use and mention</u>. [See, for example, Willard V. O. Quine, <u>Methods of Logic</u> (New York: Henry Holt and Company, Inc., 1950), pp. 37-38.]

In order to <u>mention</u> something you must <u>use</u> a symbol which names it. For example, in the sentence:

4 is an even number,

the number 4 is mentioned, but to do this a symbol for it is used. In a sentence such as the one given, there is little chance for confusing use and mention. But when the mentioned referent is itself a symbol, there is a greater chance for confusion. For example, in the sentence:

'4' is pronounced as 'fōr',

(continued on T. C. 9D)

23. Make an '8' out of wire, cut it in half the long way, and you'll get a '3' and another '3'.

24. He put '2' next to '14' and got '142'.

25. To multiply by 10 you add a '0'.
[This is intended to sound something like the rule learned in grade school for multiplying by ten. Of course, 'add' is used in the non-technical sense. Some people prefer to say 'affix' or 'annex'. Of course, even here, one can get into trouble, for if we have the number 9.25 and merely affix a '0' to the numeral '9.25', we have not multiplied 9.25 by 10.]

26. Move the decimal point in '222.235' 2 places to the left.
[Be sure to note that symbols have decimal points, but numbers do not.]

* * *

The statements in the box are the ground rules for the way we talk about symbols. We think it is necessary to prepare students for some of the strange-looking expressions and punctuation they will encounter later in the text. We are giving so much attention to this matter of semantics because we think semantic difficulties constitute a large part of the students' troubles in learning mathematics, from the first grade on up. For example, if a student has been trained so that he thinks that when he writes a name for the number 7, he is also writing the number 7 itself, he is as badly off as a student who thinks that he is putting an elephant on the blackboard when he writes the word 'elephant' on the blackboard.

* * *

(continued on T. C. 9C)

14. He erased the '5' and put a '4' in its place.

15. Bill erased the 'x' and put a '7' in its place.

16. Bill replaced 'y' by '5'.

Note: The exercises above are very important; they pave the the way for the treatment of letters in Unit 2. You may want to have the students make up a few more exercises stressing that when you substitute, you substitute one symbol for another.

17. I have put '6 ÷ 2' in the blank space.

18. A '3' appeared in the third space.

19. We cross out the 'a' and write '5' next to the crossed-out 'a'.

20. The man saw '62.5' on the meter.
[You can't see a number any more than you can touch it. You only see its name.]

21. The expressions '$\frac{6+2}{4}$' and '$\frac{8}{4}$' and '2' are expressions for the same number, but '2' is the simplest of these expressions.
[We are using the word 'expression' to mean any collection of symbols. This exercise is an early attempt to keep that old bugaboo of algebra students and teachers from arising--the simplification of expressions. We hit the notion of simplification many times in this unit, especially on page 1-71. It gets a heavy treatment in Unit 2.]

22. You can find the sum of 2 and 3 but you can't find the sum of '2' and '3'.

(continued on T. C. 9B)

14. He erased the 5 and put a 4 in its place.
15. Bill erased the x and put a 7 in its place.
16. Bill replaced y by 5.
17. I have put $6 \div 2$ in the blank space.
18. A 3 appeared in the third space.
19. We cross out the a and write 5 next to the crossed-out a.
20. The man saw 62.5 on the meter.
21. The expressions $\frac{6+2}{4}$ and $\frac{8}{4}$ and 2 are expressions for the same number, but 2 is the simplest of these expressions.
22. You can find the sum of 2 and 3 but you can't find the sum of 2 and 3.
23. Make an 8 out of wire, cut it in half the long way, and you'll get 3 and another 3.
24. He put 2 next to 14 and got 142.
25. To multiply by 10 you add a 0.
26. Move the decimal point in 222.235 2 places to the left.

> Throughout the rest of this book, when we talk about symbols and words, we shall use one of the following signals:
>
> 1. Put single quotation marks around the words or symbols.
> 2. Display the words or symbols on a separate line.
> 3. Underline words or symbols used for the first time.

1.03 <u>Distance and directions</u>.--In your early work in mathematics you learned how to use <u>whole numbers</u> (such as 7, 51, and 326) and <u>fractional numbers</u> (such as $3\frac{1}{2}$, 28% and .04) in solving problems. In this unit you will study another kind of number and learn how to use these numbers in solving problems.

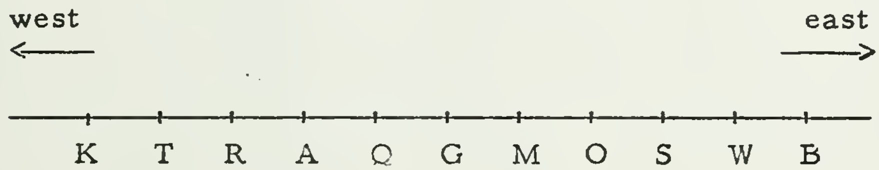
Our development of directed numbers is almost completely an intuitive one. In this sense, students learn how to use directed numbers just as they learned to use the numbers of arithmetic. We do not give a thorough treatment of directed numbers from the point of view of the mathematician. However, we have refrained as much as possible from establishing ideas which would have to be unlearned at a later time. In our present treatment, we do not tell students what a directed number is, just as he has not been told what a number of arithmetic is. Instead, we give him physical interpretations of the numerals of directed numbers, and trust that he will build for himself an understanding of the abstractions which are directed numbers. This understanding is developed as the student works with the interpretations. [Thurston, op. cit., deals with the question, "What are directed numbers?"]

* * *

You will note that we do not use a number line in our present development until page 1-62. The number line development of the earlier editions contributed to the formation of certain misunderstandings. Directed numbers can be taught much more easily without using the concept of fixed origin. That is, when a directed number symbol is interpreted as meaning a trip of a certain number of units in a certain direction, the need for an origin vanishes.

TAKING A TRIP

Imagine an east-west road which has markers placed one mile apart at the side of the road. The markers are labeled with letter symbols. If you ride a bicycle from A to G, you



make a trip of 2 miles to the *east*. If you ride from S to B, you make another trip of 2 miles to the east. Name three other trips of 2 miles to the east.

Trips from Q to R or from S to M or from A to T are trips of 2 miles to the *west*. State three other trips of 2 miles to the west.

Now, these trips of 2 miles to the east and 2 miles to the west are alike in one important way. Each trip covers a *distance* of 2 miles. But, the trips to the east differ from the trips to the west in another important way. The trips to the east are made in a *direction* opposite to that of the trips to the west. You are going to learn about numbers which tell both

the distance covered in a trip
and
the direction of a trip.

The numbers are new to you; they are called *directed numbers*.

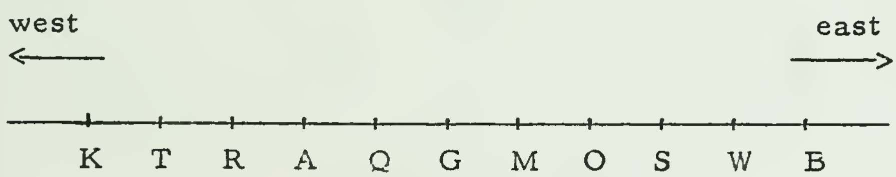
Let us see how these numbers are used. Consider the trips of 2 miles to the east. We use the symbol '+2' as a numeral for a directed number which corresponds to all of these trips. We can use the symbol '-2' as a numeral for a directed number which corresponds to all of the trips of 2 miles to the west. The positive number +2 and the negative number -2 correspond to trips made in opposite directions but covering equal distances.

Take care to see that students do not think of the number -2 as "the number 2 with a ' - ' attached to the number". If necessary, you can tell your students that such thinking is similar to thinking that Maryland is Mary with 'land' attached to her! On the other hand, a numeral for -2 is the numeral '2' with a ' - ' prefixed to it.

> The symbol '+2' is a name for a directed number. Another name for this directed number is 'positive two'. The number +2 is also called a positive number.
>
> The symbol '-2' is a name for a directed number. Another name for this directed number is 'negative two'. The number -2 is also called a negative number.

A pair of directed numbers like +2 and -2 are called opposite numbers. We say that +2 is the opposite of -2 and that -2 is the opposite of +2. We also say that a positive number corresponds to trips made in the positive direction, while a negative number corresponds to trips made in the negative direction. When we used +2 to correspond to trips of 2 miles to the east, we had decided that the east direction would be the positive direction and that the west direction would be the negative direction. (We could just as well have chosen the west direction as the positive direction.)

If you are told that someone has taken a trip of +4 miles, you know that he has traveled 4 miles to the east. Perhaps he has traveled from T to G, or from M to B, or between



other points. From the statement 'a trip of +4 miles' you can tell that the starting point and the endpoint of the trip are 4 miles apart and that the endpoint is east of the starting point. If you also know that the starting point was A, you can easily tell that the endpoint was O.

Suppose you are told that a trip of -3 miles was made on this road. The number -3 corresponds to trips of 3 miles

Notice that any correct answer to Exercise 4 of Part B is also a correct answer to Exercise 5, and that in such trips the distance covered is 0 miles. This strongly suggests that -0 is the same number as +0 as, of course, is the case. In other words, -0 = +0. Naturally, neither -0 nor +0 is 0, any more than the directed number +5 is the number 5 of arithmetic. In Section 1.08 these questions are treated in considerable detail. Until we reach that point we shall write '+0' or '-0' rather than '0' when we are working with directed numbers.

Of course, students will soon learn that +0 and -0 are equal. But this fact does not allow us to say that +0 and -0 and 0 are all equal. If this sounds strange to you, see the COMMENTARY for Section 1.08. However, now is not the time to make an issue of this in class. When students discover that +0 = -0, this discovery should be accepted by the entire class as a not very startling fact.

to the west on this road. Examples of such trips are those from W to M, and from G to R, and from A to K. If the starting point of such a trip was, say, M, then the endpoint was A; if the endpoint was T, then the starting point was Q. In every case, the starting point and the endpoints are 3 miles apart and the endpoint is west of the starting point.

EXERCISES

A. Give the directed number which corresponds to each of the following trips made along an east-west road if the east direction is the positive direction and the markers are 1 mile apart.

negative (west) $\longleftarrow$ positive (east) $\longrightarrow$

F R K L J H P X A Z M T

1. R to J	2. X to T	3. M to A
4. L to F	5. P to X	6. X to P
7. H to K	8. R to P	9. T to P

B. Use the diagram in Part A and give 5 trips, each of which corresponds to the given directed number.

1. +6
2. -5
3. +5
4. +0
5. -0

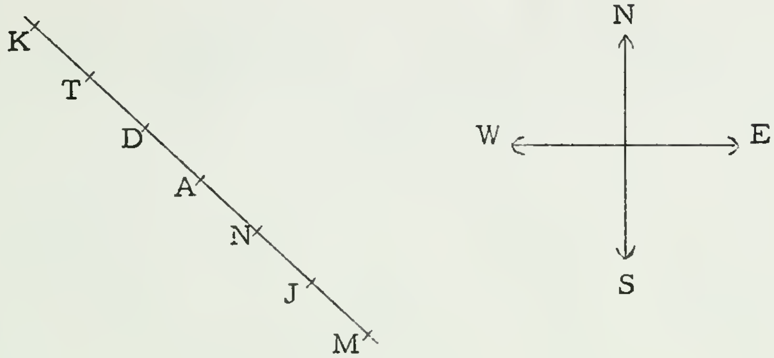
C. Use the diagram in Part A and complete the following table. (The first exercise has been completed.)

		Start	Finish
1.	+7	F	X
2.	-3	A	
3.	+2	X	
4.	-5	P	
5.	-4	Z	

		Start	Finish
6.	-1	A	
7.	+4		H
8.	-6		L
9.	-2		L
10.	+8		M

Exercise 3 of Part D requires some manipulatory skill with fractions. So does Exercise 4 at the top of the next page. Here is a place where you can do some "painless" remedial work. The diagram is a convenient physical representation of the "whole-part business" stressed in earlier grades. You and the students can make problems ad infinitum in which the unit of distance is $\frac{2}{3}$, or 0.6, or $5\frac{1}{2}$, etc. Naturally, we don't expect complete mastery of fractional numbers as a result of this type of exercise. There will be ample opportunities to come.

D. Suppose instead of an east-west road, we have a road which is illustrated in the following map.



The letters stand for markers placed 1 mile apart. Trips can be made in either of two directions along this road. That is, a trip can be made in the same direction as one from K to M or in the opposite direction as from M to K. Choose one of these directions as the positive direction. Then trips made in the opposite direction are made in the negative direction. Once you have made your choice of the positive direction, stick to it throughout these exercises.

1. Give a directed number which corresponds to each of the following trips.

   (a) T to A (b) J to A (c) K to N

   (d) A to T (e) A to J (f) N to K

2. Complete the following table.

		Start	Finish
1.	+2	D	
2.	-2		D
3.	+1	T	
4.	-1		T

		Start	Finish
5.	+3	A	
6.	-3		A
7.	+1	N	
8.	-1	N	

3. Suppose the markers are $\frac{1}{2}$ mile apart instead of 1 mile. Give a directed number which corresponds to the number of miles in each of the following trips.

   (a) T to A (b) M to N (c) N to K

   (d) N to T (e) K to J (f) M to K

In Part E point out that the opposite of a number can be named in many ways. For example, the student should recognize in particular that '-(-6)' and '+6' are names for the same number, that is, -(-6) = +6. Avoid having students say: "To find the opposite of a number you change its sign." The number -2 does not have a sign; the numeral '-2' does have a sign. Note the relation between the exercises in Parts E and F. Keep students from saying aloud things like "Two minuses make a plus."

* * *

Opposite numbers come up again on pages 1-55 and 1-56 and in Unit 2. But now is the time to elicit from them the important principle that each directed number has one and only one opposite.

* * *

In adding directed numbers do not encourage students to verbalize a rule. Just as the first grader knows that to add 5 and 3 he should "take" 5 apples and 3 apples and "combine" the apples, your ninth grader should learn to add +5 and +3 by combining trips. We want students to have a lot of intuitive feeling for adding directed numbers before they seek nicely-worded rules. It may be the case that some student will formulate a rule after working for a while with directed numbers, or after talking to a more advanced student. In any event, you may feel forced to state the rule for the entire class. Don't do it! The danger in stating a rule of this sort is that an unsuccessful student turns to the rule for help in adding when he should turn to the interpretation for this help. The rule stated in conventional texts for adding the directed numbers, say, +7 and -3, is only a formal device for obtaining a simple name for +7 + (-3). The rule places too much emphasis upon manipulation of names. At this stage of learning, we want the stress placed on the interpretation of the symbols. There will be plenty of rules in Unit 2.

4. Suppose the markers at K and M are 8 miles apart and all of the markers are evenly spaced. Give a directed number corresponding to each of the following trips.

   (a) K to M (b) M to K (c) A to T

   (d) D to J (e) K to A (f) N to T

E. Give the opposite of each of the directed numbers.

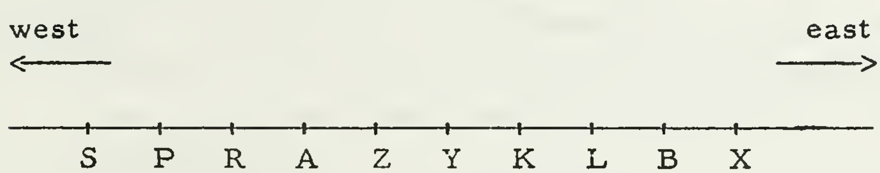
1. -6	2. $+7$	3. $-10\frac{1}{2}$
4. $-\frac{1}{3}$	5. $+6.2$	6. -9.75

F. Give the <u>opposite</u> <u>of</u> <u>the</u> <u>opposite</u> of each of the directed numbers.

1. -6	2. $+7$	3. $-10\frac{1}{2}$
4. $-\frac{1}{3}$	5. $+6.2$	6. -9.75

1.04 <u>Adding directed numbers</u>.--Since whole numbers and fractional numbers can be added, subtracted, multiplied, and divided, it is natural to ask whether we can also do these things with directed numbers.

When you first learned about whole numbers, you thought of them as corresponding to groups of things. Then you learned to add these numbers by combining the groups. Now, you think of a directed number as corresponding to a trip and you will learn to add directed numbers by combining trips.



We select the east direction as the positive direction and specify that the markers are 1 unit of distance apart.

<u>Example 1</u>. Two trips are made, one following another. The first is made from A to Y and the second from Y to B. Give a single trip which has the same result as that of the combined trips.

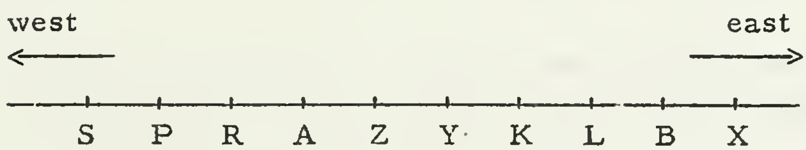
(Solution on next page)

Solution. The two trips take the traveler from A to B. So, a single trip from A to B has the same result as a trip from A to Y followed by a trip from Y to B.

The trip from A to Y corresponds to the directed number +2, and the trip from Y to B corresponds to the directed number +3. Now, it is reasonable to think combining these two trips corresponds to adding the directed numbers +2 and +3. The single trip from A to B corresponds to the directed number +5. So, we shall say that

$$\boxed{+2 + (+3) = +5}$$

(Note that in the boxed statement we have placed parentheses around '+3' to avoid confusing the '+' in '+3' with the sign of addition.)



Example 2. Two trips are made, one following the other. The first is made from L to Z and the second is made from Z to S. Give a single trip which has the same result as that of the combined trips.

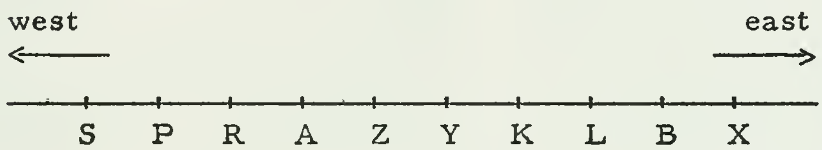
Solution. The result of the combined trips is to take the traveler from L to S. Therefore, a trip from L to S is a single trip which has the same result as the combined trips.

The trip from L to Z corresponds to the directed number -3 and the trip from Z to S corresponds to the directed number -4. The single trip from L to S corresponds to the directed number -7. If adding directed numbers corresponds to combining trips, then we can write:

$$\boxed{-3 + (-4) = -7}$$

Example 3. Give a single trip which has the same result as combining two trips, the first P to K and the second from K to Z.

Solution. We are concerned here with the starting point and the final endpoint. When we ask for a single trip which gives the same result as combining two trips, we are seeking the shortest trip which begins at the same starting point as the first of the combined trips and finishes at the same endpoint as the second of the combined trips. Therefore, combining a trip from P to K with a second trip from K to Z gives the same result as a single trip from P to Z.



Since a trip from P to K corresponds to +5, and a trip from K to Z corresponds to -2, and a trip from P to Z corresponds to +3, we write:

$$\boxed{+5 + (-2) = +3}$$

Example 4. Give a single trip which has the same result as combining two trips, the first from B to P and the second from P to Y.

Solution. The single trip should take the traveler from B to Y. The trip from B to P corresponds to -7, the trip from P to Y corresponds to +4, and the single trip from B to Y corresponds to -3. Thus,

$$\boxed{-7 + (+4) = -3}$$

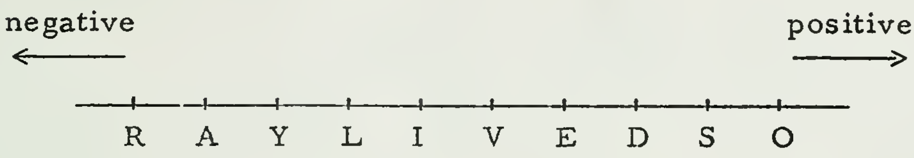
Exercises 5, 6, 9, and 10 describe trips which correspond to '+0' or '-0'. Naturally, the students can use either symbol, but at this stage they should not use the symbol '0'.

* * *

Raise the question: Suppose that instead of starting the trip from D, the trip had been started from some other point. Would you get a different sum?

EXERCISES

A. In each of the following exercises combine the two given trips using the diagram. Then make a statement about



the addition of directed numbers corresponding to the trips.

Sample. From A to D and from D to E

Solution. Combining these trips gives a trip from A to E. From A to D corresponds to +6. From D to E corresponds to -1. From A to E corresponds to +5. The corresponding addition statement is:

$$+6 + (-1) = +5$$

1. From Y to V and from V to S.
2. From A to D and from D to I.
3. From S to E and from E to R.
4. From D to R and from R to E.
5. From E to A and from A to A.
6. From L to L and from L to V.
7. From V to Y and from Y to D.
8. From S to Y and from Y to R.
9. From D to E and from E to D.
10. From S to S and from S to S.

B. Add the two directed numbers listed in each of the following exercises by selecting trips which correspond to each number and combining these trips.

Sample. -6, +4

Solution. For a trip corresponding to -6 we select, say, the trip from D to A. A trip starting at A which corresponds to +4 is the trip from A to V. Combining these trips, we have the trip from D to V which corresponds to -2. So, we write:

$$-6 + (+4) = -2$$

In Part C the student should come away with the feeling that this process always gives either +0 or -0. The verbalization of a rule is not essential however, You may be able to check whether the student has the generalization or not by asking him for the sum of, say, +4,567,809 and -4,567,809.

* * *

In Part D watch carefully in order to tell when the students are operating successfully. Again, avoid stating a rule. Students need to become fairly proficient in adding directed numbers. Thus, you may have to provide more exercises of this type for the less able student. You may want to do some further remedial work in fractions also [Use a "part-whole" diagram for Exercises 29 and 30.], but it is probably inadvisable to belabor a student with the arithmetic of fractional numbers while he is struggling to obtain a procedure for adding directed numbers.

* * *

Note the use of the word 'listed' in the instruction for Part D. Exercises are statements or other kinds of marks on paper. Numbers do not occur in exercises, although numerals do. However, numbers can be listed in exercises, because 'listing' means 'writing names for'. When someone holds up a sheet of paper and says, "I have here a <u>list</u> of people", he means, of course, that the names of people are written on the sheet of paper.

1. +2, +5
2. -4, -4
3. -3, +7
4. -7, +3
5. $+2\frac{1}{2}$, $+3\frac{1}{2}$
6. -1, +1
7. $+3\frac{1}{3}$, -0
8. +127, -127
9. +4, $-3\frac{1}{5}$
10. $-\frac{1}{3}$, $-\frac{1}{2}$

C. For each of the following exercises give the opposite of the directed number. Then find the sum of the directed number and its opposite.

1. +5
2. -6
3. $+2\frac{1}{2}$
4. -6.5
5. +1
6. +0

D. Now that you have learned to add directed numbers using a lettered diagram, you will be able to add directed numbers by picturing such a diagram in your mind. Add the directed numbers listed in each of the following exercises without looking at a diagram.

1. -3, -5
2. +1, +3
3. +10, -10
4. +2, -1
5. -2, +1
6. +7, -3
7. +3, -7
8. -8, -2
9. +2, +9
10. -6, +7
11. -9, +9
12. +0, -5
13. +2, -0
14. -3, -0
15. +21, -15
16. -12, +13
17. -32, -42
18. +17, +19
19. -181, +75
20. +181, -75
21. -1000, +2000
22. -101, +203
23. $+\frac{1}{5}$, $-\frac{3}{5}$
24. $-\frac{2}{7}$, $-\frac{3}{7}$
25. $-\frac{1}{4}$, $+\frac{1}{2}$
26. $-1\frac{1}{2}$, $-3\frac{1}{4}$
27. +4.6, -3.2
28. +0, -7.5
29. $-9\frac{1}{6}$, $+3\frac{1}{3}$
30. +6.25, $-7\frac{1}{5}$

Illustrations I and II attempt to give the student more interpretations of directed number symbols. In other words, we are extending the intuitive notion of a trip.

* * *

Illustration I does not contain a direct example of adding directed numbers. We are using directed numbers merely in a descriptive way. It might be instructive to have students try to add directed numbers in helping them interpret the table. For example, you can describe the change from the number in 1932 to the number in 1934 by adding the two differences.

1.05 Using directed numbers.--You have learned that a directed number may correspond to a trip. What are the characteristics of a trip that are used to obtain the corresponding directed number? You will recall from working with directed numbers and trips that these characteristics are:

1. change by a certain amount.
2. change in either of two opposite directions.

We can use directed numbers in any other situation in which measurable changes occur in either of two opposite directions. Let us look at some situations where directed numbers apply.

ILLUSTRATION I

In social studies class the teacher handed out a quiz and a copy of a table with the following information:

TOTAL IMMIGRATION TO U. S.
1932 - 1940

1932	35,576
1933	23,068
1934	29,470
1935	34,956
1936	36,329
1937	50,244
1938	67,895
1939	82,998
1940	70,756

The quiz contained many questions like these:

Between what two successive years was there the greatest decrease in immigration?

(continued on next page)

Between what two successive years was there the largest increase in immigration?

Between what two successive years was there the smallest increase in immigration?

Answer the questions above before reading further.

One of the girls in the class completed the test much sooner than anyone else. Here is what she had done to her table:

TOTAL IMMIGRATION TO U. S.
1932 - 1940

1932	35,576	
		−12,508
1933	23,068	
		+6,402
1934	29,470	
		+5,486
1935	34,956	
		+1,373
1936	36,329	
		+13,915
1937	50,244	
		+17,651
1938	67,895	
		+15,103
1939	82,998	
		−12,242
1940	70,756	

How did she answer the three questions given above using her table?

Explain how directed numbers can be applied to this table, that is, tell what are "trips" in this case. How do you tell the amount of a change? How do you tell the direction of a change?

ILLUSTRATION II

Bill's father gave him $3 to start and operate a flower business for one week. His father told him to use the $3 to buy flowers the first day and to sell as much as he could each day. He also told him to use all of the money he collected on one day to buy flowers the next morning. Although spending all of his money each morning might not be the best business procedure, his father wanted to see how far up he could "run" the $3.

Here is a record of his week's business:

	Expenses	Sales	Profit-Loss
Monday	$3.00	$4.00	+1.00
Tuesday	4.00	5.20	+1.20
Wednesday	5.20	4.80	-0.40
Thursday	4.80	4.80	+0.00
Friday	4.80	4.70	-0.10
Saturday	4.70	6.80	+2.10

Explain how directed numbers correspond to the outcome (profit or loss) of each day's business.

How could addition of directed numbers be used with this table? Suppose, for example, you add +1.20 (Tuesday's outcome) and -0.40 (Wednesday's outcome).

$$+1.20 + (-0.40) = +.80$$

Does the directed number +.80 correspond to the combined outcome of two days? In other words, does +.80 correspond to the result of two "trips"?

EXERCISES

A. Refer to Illustration II above. What is the combined outcome of business on

1. Wednesday and Thursday?
2. Monday and Tuesday?
3. Thursday and Friday?

B. Refer again to Illustration II.

Sample. What is the combined outcome of business on Monday and Wednesday?

Solution. Let us add the directed numbers corresponding to each outcome:

$$+1.00 + (^-0.40) = +0.60$$

Does this sum correspond to the combined outcome? Our usual check is to make a single "trip" from the "starting point" on Monday to the "endpoint" on Wednesday. But we note that in this case the "endpoint" on Monday is not the "starting point" on Wednesday. So, the single "trip" idea cannot be used.

Here is another way of checking to see whether the sum +0.60 corresponds to the combined outcome:

Combine expenses for the two days and combine sales for the two days.

Expenses comb.	3.00 + 5.20	=	8.20
Sales comb.	4.00 + 4.80	=	8.80

The combined expenses, \$8.20, can be considered as expenses for a combined "day". Similarly, the combined sales can be considered as sales for a combined "day".

	Expenses	Sales	Outcome
Mon. - Wed. comb	\$8.20	\$8.80	+0.60

The "outcome" +0.60 checks with the sum obtained by adding the directed numbers corresponding to the individual outcomes.

In considering the exercises in Part B, one might also ask:

5. How much profit did Bill make during the week?

In Part C, be sure the student does (b) for each exercise. Some teachers reported that their students overlooked this item of the instructions.

In each of the following exercises find the combined outcome by adding directed numbers. Check your answer by using the method of checking shown above.

1. Wednesday and Friday
2. Tuesday and Thursday
3. Wednesday and Saturday
4. Monday, Wednesday, and Friday

C. Each of the phrases (starting at the top of page 1-24) gives an amount of change and a direction of change.

(a) Write a phrase corresponding to a change in the opposite direction from the given direction, but of the same amount.

(b) Choose one of the directions for the positive direction and

(c) give the directed number which corresponds to the phrase given in the exercise and

(d) give the directed number which corresponds to the phrase you wrote in (a).

<u>Sample.</u> 2 hours late

<u>Solution.</u> (a) 2 hours early

(b) choose "late" as the positive direction

(c) +2 (2 hours late)

(d) -2 (2 hours early)

1. $1\frac{1}{2}$ miles above sea level
2. 35° below zero
3. 2 hours before noon
4. 12 days ago
5. 5 pounds heavier
6. 35 miles north
7. 3 yards gained
8. $15 in debt
9. an increase of $2\frac{1}{4}$ points
10. 7 pounds underweight
11. 3 points above average
12. a loss of 7 yards
13. 10 years hence
14. 25 feet underground
15. 2 years older
16. 15 points to the good
17. a loss of 35 cents
18. a credit of $27
19. a price rise of $6
20. a shortage of 12 items

D. Use directed numbers to answer the following questions.

1. Bill made 3 dollars profit the first day of business, lost 6 dollars the second day, and made 5 dollars profit the third day. What was his standing at the end of the third day?

2. Ed made $6.80 profit the first day of business, made $2.55 profit the second day, lost $5.42 the third day, and made $1.53 the fourth day. What was his standing at the end of the fourth day?

3. Zabranchburg High's football team gained 3 yards the first down, lost 4 yards the second down, gained 5 yards the third down, and gained 7 yards the fourth down. Did they make a first down?

4. John and Fred are playing a game. John wins 3 points in the first round, loses 4 points in the second

Betty passes Jane at the 'R' signboard which points west and reads, "7 miles to Fish Hook". She continues hiking east until she reaches a certain apple tree along the road. She turns here (after hastily picking an apple to munch on the way home!) and starts back to Q. On the way she meets Jane at the 'W' signboard. It points west and reads, "10 miles to Fish Hook".

If both Betty and Jane are hiking at steady rates, what is the distance of the apple tree from the village of Fish Hook? What direction is the tree from the village?

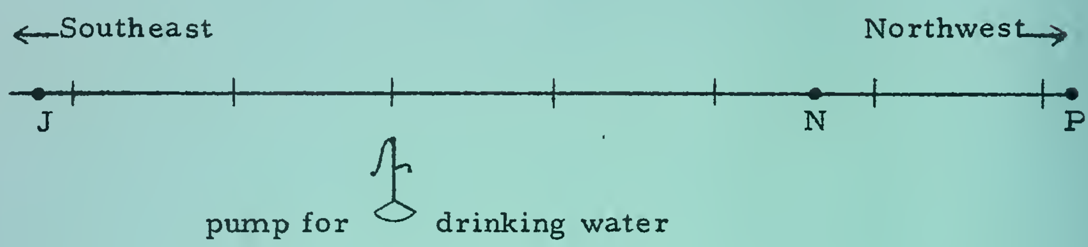
at 10:00 a.m. to go visit Bill. When Nick arrives at Bill's home and finds he isn't there, he has a hunch that Bill may have gone to the playground, so he goes on toward it. Meanwhile, Bill hasn't found anything of interest at the playground, so he has started on to the swimming pool, which is 8 blocks farther. When Nick can't see Bill at the playground, he turns around and starts home; however, when he gets there he is still wanting to talk to Bill, and decides to return to Bill's home. But Bill is on his way toward Nick's home, since he saw none of his friends at the pool. If Bill and Nick have walked at the same rate (and you disregard any time lost while they look around at playground and pool), can you discover how far from Nick's home the two boys finally meet?

3. Ann starts from Sally's home and bicycles north. At the same time, Carol starts from her own home, which is 22 blocks south of Sally's, and is also bicycling north. Ann travels only 4 blocks when she turns around and starts south. Carol is 18 blocks north of Sally's home when she turns back; at that time Ann is midway between the point where Carol turns back and Carol's home. When Carol gets home and does not see Ann, she starts north again and travels until she meets her. How far from Sally's home do they meet? In what direction from Sally's home is the meeting point?

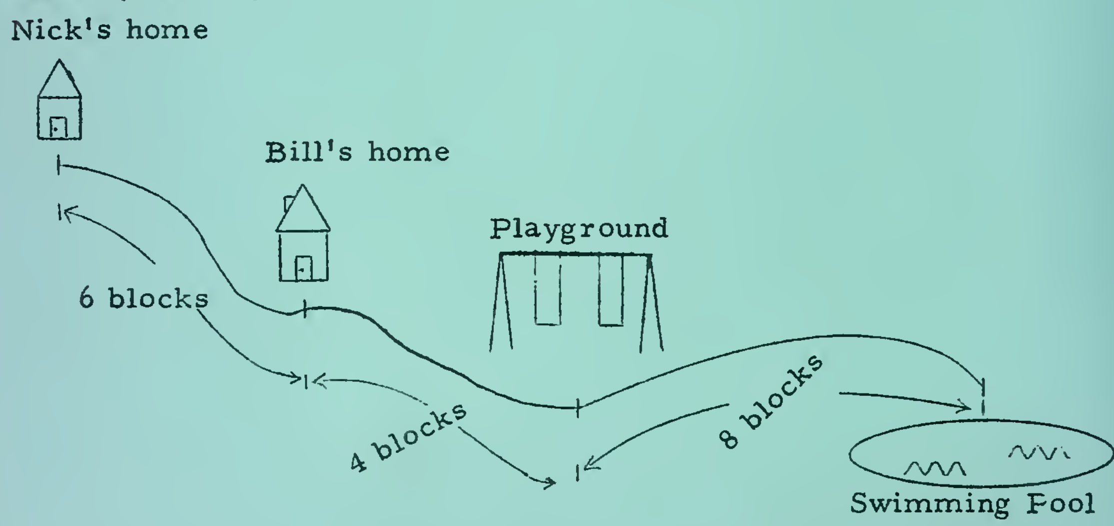
4. Jane and Betty both start hiking at the same time, and both travel in an easterly direction. Jane starts at M, where a sign post pointing west reads "4 miles to Fish Hook". Betty starts at Q; the sign board there points east and reads, "8 miles to Fish Hook".

(continued on T. C. 25D)

Answer the following questions. (You may want to make a diagram on the sketch.)

1. 

On a hiker's trail, John starts from P which is 21 units northwest of the spot where pure drinking water may be obtained, and walks toward the pump. Bob starts at J (which is 11 units southeast of the pump) at the same time, and also hikes toward it. They pass each other at N, which is 13 units northwest of the pump, and continue on their way. However, Bob becomes thirsty and turns back at P, to hike to the pump, and then on to J. Does Bob overtake John before he arrives at the pump? How far from the pump is John when Bob overtakes him? [The boys hike at steady rates.]

2. 

Bill leaves home at 10:00 a.m. to walk to the playground which is 4 blocks away. His friend Nick leaves his home

(continued on T. C. 25C)

Exercises 5 and 6 are optional exercises as indicated by the star at the left of the exercise numeral. Optional problems are usually fairly difficult and somewhat off the main track. Unless there is a large number of students in the class who can get excited over an optional problem, it is probably best not to spend much class time in discussing them. If your students are interested, there are further questions that you might ask in regard to Exercises 5 and 6. For Exercise 5, you might ask, "If directed numbers are used to indicate the trips up and down, what will their sum be?" For Exercise 6, one can make up other variations of the problem . For example,

a) Suppose one travels twice as fast as the other?

b) Suppose the two boys do not start from the same point?

* * *

The students will learn to multiply directed numbers by considering a pump and a tank. The exploration exercises are to familiarize the student with time-rate problems before considering directed numbers.

* * *

If your students are really interested in problems such as Exercises 5 and 6, here are others you might use.

(continued on T. C. 25B)

round, and wins 5 points in the third round. What is his score at the end of the third round?

☆5. A department store has 6 floors above ground level and 2 floors below ground level. The first floor above ground level is called 'mezzanine', the next floor above the mezzanine is called 'first floor', the floor next above the first floor is called 'second floor', etc. The first floor below the ground floor is called 'first basement' and the floor below that one is called 'second basement'. An operator makes the following trip: ground floor to mezzanine to ground floor to first floor to third floor to fourth floor to ground floor to first basement to ground floor to second basement to ground floor. If the floors are 17 feet apart, how many feet does he travel during the trip?

☆6. Two cyclists start from home at the same time. John travels 4 miles east, then 2 miles west, then 3 miles east. He then travels west until he meets Walt. Walt starts by traveling 3 miles west, then 1 mile east, then 3 miles west, then east until he meets John. If both cyclists travel at the same speed, how far from home do they meet? In which direction must they travel if they head directly for home together? How many miles has each cyclist traveled by the time they reach home?

1.06 <u>Multiplying directed numbers</u>.--We shall learn to multiply directed numbers by looking at problems to which directed numbers apply. But first let us review a problem which you can solve by multiplying numbers of arithmetic.

EXPLORATION EXERCISES

<u>A</u>. Suppose that a pump <u>fills</u> a tank with water at a rate of 3 gallons per minute. What is the <u>increase</u> (gallons) in the volume of water in the tank

1. 1 minute <u>from now</u>?
2. 4 minutes from now?
3. $10\frac{1}{2}$ minutes from now?
4. 0 minutes from now?

(2) Interpret '-7' as follows: A movie camera takes a 7-minute picture of the pump and tank. The film is developed, put into a projector, and the projector is run backwards (-).

(3) Finally, interpret the symbol '(+2) × (-7)' as meaning the apparent change in volume of the water in the tank as shown on the screen. A decrease in volume corresponds to a negative number; an increase corresponds to a positive number.

(4) With this interpretation of '(+2) × (-7)', we find that '(+2) × (-7)' is another name for the directed number -14. So, (+2) × (-7) = -14.

***

Let the students reach agreement about the "directions" of the various elements in the tank model. Note that the choices of direction are clearly arbitrary; the students should be told that we are selecting directions which will give results in accord with those obtained by strictly mathematical methods.

***

Mr. Steele reported that this exposition on multiplication of signed numbers was very useful in his conventional elementary algebra classes. Naturally, you are free to use the approach with your conventional courses if you want to. We realize that such an action on your part will tend to contaminate our "controls" but we are happy to make this sacrifice for the good of mathematics teaching. Let us know how the approach works in your conventional classes.

Again, as in the case of addition, we use an interpretation of numerals for directed numbers in order to teach multiplication.

Here is a description of what was done in addition, and a similar description of what we do in multiplication.

Addition

How shall we interpret the symbol:

$$(+2) + (-7)$$

(1) First, interpret the symbol '+2' as meaning a trip of 2 units in one of two opposite directions. The direction you choose is then called positive.

(2) Interpret '-7' as meaning a trip of 7 units in the negative direction.

(3) Finally, interpret the symbol '(+2) + (-7)' as meaning a trip you would take to get you from the starting point of the +2 trip to the terminal point of the second leg (-7) of the two-leg trip.

(4) With this interpretation of '(+2) + (-7)', we find that '(+2) + (-7) is another name for the directed number -5. So,

$$+2 + (-7) = -5.$$

Multiplication

How shall we interpret the symbol:

$$(+2) \times (-7)$$

(1) First, interpret the symbol '+2' as meaning the filling (+) of a tank by a pump operating at the rate of 2 gallons per minute.

B. Suppose that the pump empties the tank at a rate of 3 gallons per minute. What is the decrease in the volume of water in the tank

1. 1 minute from now?
2. 4 minutes from now?
3. $10\frac{1}{2}$ minutes from now?
4. 0 minutes from now?

C. Suppose the pump fills the tank at a rate of 3 gallons per minute. How many less gallons of water were there in the tank

1. 1 minute ago?
2. 4 minutes ago?
3. $10\frac{1}{2}$ minutes ago?
4. 0 minutes ago?

D. Suppose the pump empties the tank at a rate of 3 gallons per minute. How many more gallons of water were there in the tank

1. 1 minute ago?
2. 4 minutes ago?
3. $10\frac{1}{2}$ minutes ago?
4. 0 minutes ago?

A MOTION PICTURE OF PUMPING

You have probably guessed that directed numbers could be used in the pump-tank problem in the Exploration Exercises. We shall now show you how directed numbers apply in that problem.

We shall let directed numbers correspond to the various changes as follows:

gal. increases in volume of water - positive numbers
gal. decreases in volume of water - negative numbers

gal. per minute, filling the tank - positive numbers
gal. per minute, emptying the tank - negative numbers

We also need to have directed numbers correspond to time in some way. You are not accustomed to thinking of time changing in either of two opposite directions. So, we shall imagine that a motion picture has been made of a pump filling

Suppose the pump is filling and that a film is being made of the process. When this film is projected with the projector running backwards, the observer sees that the pump is emptying the tank rather than filling the tank. Thus, it looks as if you are contradicting yourself when you say that an "emptying pump" is a "filling pump". Actually, it is this apparent contradiction which gives the correct rule for multiplication. If any student should raise this issue, tell him to think of a sign hanging on the pump. The sign bears either the word 'FILLING' or the word 'EMPTYING' to describe its condition while the film was being made.

* * *

The instructions for Part A are rather heavy. Be sure the students know what is going on before they plunge too deeply into the exercises which start on page 1-28.

a tank and a motion picture made of a pump emptying a tank. When we show the picture with the film running forwards (normally), then time changes correspond to positive numbers. But, if the film is run backwards, then time changes correspond to negative numbers. (Have you ever watched a comedy film in which a man seems to dive up out of the water and onto a diving board!)

Now, when the pump is <u>emptying</u> the tank and the film is run <u>forwards</u>, you can see the volume of water <u>decreasing</u>. But, when the film is run <u>backwards</u> while the pump is really <u>emptying</u>, you can see the volume of water <u>increasing</u>. What do you see happening to the volume of water (increase or decrease?) when the pump is filling the tank if the film is run forwards? If the film is run backwards?

EXERCISES

A. The table below contains problems dealing with the pump-tank-film idea. From each problem you can learn how to multiply a pair of directed numbers. We have solved the first problem for you as a sample.

In this problem you are told that a pump is filling the tank at the rate of 4 gallons per minute. Therefore '+4' is written in the column headed 'Pump'. Then, you learn from the second column that the pump has been operating for 2 minutes and that the film is running backwards. Therefore, we write '-2' in this column. Now, we ask about the change in volume of water that would be observed. Since the pump is filling the tank as indicated by +4, and the film is running backwards as indicated by -2, the volume of water appears to be decreasing. So, we actually observe from the film a decrease in volume of 8 gallons. The number -8 corresponds to this observed change. In the last row for the first problem we write the corresponding multiplication statement. Complete the table.

	Pump	Time	Observed Change in Volume
1.	Filling, 4 gallons per minute	2 minutes, running backwards	decrease of 8 gallons
	$+4$	-2	-8
	Corresponding multiplication: $+4 \times (-2) = -8$		
2.	Emptying, 4 gallons per minute	2 minutes, running forwards	
	Corresponding multiplication:		
3.	Filling, 4 gallons per minute	2 minutes, running forwards	
	Corresponding multiplication:		
4.	Emptying, 4 gallons per minute	2 minutes, running backwards	
	Corresponding multiplication:		
5.	Filling, 8 gallons per minute	3 minutes, running forwards	
	Corresponding multiplication:		

	Pump	Time	Observed Change in Volume
6.	Emptying, 8 gallons per minute	3 minutes, running forwards	
	Corresponding multiplication:		
7.	Emptying, 8 gallons per minute	3 minutes, running backwards	
	Corresponding multiplication:		

Note: In the rest of the problems you are given directed numbers and you should fill in the corresponding blanks.

8.			
	-5	-6	
	Corresponding multiplication: $-5 \times (-6) =$		

	Pump	Time	Observed Change in Volume
9.			
	+7	-3	
	Corresponding multiplication: $+7 \times (-3) =$		
10.			
	-8	+0	
	Corresponding multiplication: $-8 \times (+0) =$		
11.			
	$-6\frac{1}{2}$	-4	
	Corresponding multiplication: $-6\frac{1}{2} \times (-4) =$		

B. Multiply the directed numbers listed in each exercise. Use the pump-tank-film idea as long as it helps you. You should develop for yourself a system for multiplying directed numbers rapidly.

1. +5, +2
2. +6, +3
3. $+8\frac{1}{2}$, +8
4. $+9\frac{1}{3}$, +6
5. +6, -2
6. -2, +6
7. -5, +7
8. +8, -8
9. -9, +10
10. +12, -10
11. -7, -8
12. -15, -3
13. -1, -1
14. -8, -12

Again, we want students to make their own rules for multiplication and not feel compelled to state the rules in class.

* * *

Exercises 29 and 30 provide another opportunity for remedial work with fractions.

* * *

The "book sale" development is an illustration of principle (I) mentioned on page A on the Introduction to the COMMENTARY. We are trying to build toward a mathematical idea by using a totally fanciful situation which is completely within the students' imagination. We take a lot of pains both in our development and in the accompanying exercises in order to give the students experiences which will prepare them in a rather subtle manner for the idea itself. If we stated the idea bluntly at the outset, we are sure the students would reject it.

The idea is that the numbers of arithmetic (the unsigned reals) are not identical with the positive directed numbers (the positive reals). The nature of the difference cannot be fully appreciated until various types of numbers have been described [see Thurston, <u>op. cit</u>.]. However, the reason that one is tempted to say that the numbers of arithmetic and the positive directed numbers are the same is that the numbers in both systems "act" very much alike. Pages 1-30 through 1-39 embody an attempt to spell out what we mean by 'act very much alike'. A mathematician says that the numbers of arithmetic and the positive directed numbers are <u>isomorphic</u> with respect to addition and multiplication. The tables illustrate the intuitive idea of isomorphism with respect to one or more operations.

15. +7, -0
16. -0, -6
17. +0, -0
18. -12, +0
19. -16, $-\frac{1}{4}$
20. -100, $+2\frac{1}{2}$
21. +7, -21
22. -16, -16
23. +47, -58
24. +27, -65
25. +705, +15
26. -86, -75
27. -1.83, -1.81
28. +9.65, -7.48
29. $+15\frac{3}{4}$, $-24\frac{1}{4}$
30. $-31\frac{1}{3}$, $-86\frac{2}{3}$

1.07 A book sale.--Walt Barnes was hired one Saturday by the owner of a small book store in his neighborhood. The store was having a "clean-out" sale in both the fiction and the non-fiction departments. Here is how the sale operated.

A customer would make selections in one department and the clerk in that department would write out a sales slip showing the total cost of the purchase in that department. Then the customer might make selections in the other department in which case a clerk would record on another sales slip the total purchase in that department. If the customer had bought books in both departments, he would go, next, to Walt. Walt's job was to add the two amounts and make a combined bill which the customer took to the cashier.

Just before noon business became brisk and Walt was swamped. He was good at arithmetic but not good enough for that sale! A long line built up waiting for Walt to make his computations. The owner became upset because his customers were unhappy about the long wait. So he walked to Walt's table to try to find the difficulty. Walt's table was covered with scratch paper. Here is what his scratch pad looked like:

The last sentence in the bracketed paragraph is very important. The saleslips do not tell the number of books purchased; they tell only the total cost in each department.

$$\begin{array}{r} 3.56 \\ 6.23 \\ \hline 9.79 \end{array} \qquad \begin{array}{r} 5.34 \\ 10.68 \\ \hline 16.02 \end{array} \qquad \begin{array}{r} 9.79 \\ 7.12 \\ \hline 16.91 \end{array} \qquad \begin{array}{r} 2.67 \\ 8.90 \\ \hline 11.57 \end{array}$$

$$\begin{array}{r} 6.23 \\ 10.68 \\ \hline \cancel{16.81} \\ 16.91 \end{array} \qquad \begin{array}{r} .89 \\ 11.57 \\ \hline 12.46 \end{array} \qquad \begin{array}{r} 3.56 \\ 9.79 \\ \hline 13.35 \end{array} \qquad \begin{array}{r} 6.23 \\ 6.23 \\ \hline 12.46 \end{array}$$

$$\begin{array}{r} 14.24 \\ 1.78 \\ \hline 16.02 \end{array} \qquad \begin{array}{r} 8.90 \\ 5.34 \\ \hline \cancel{14.14} \\ 14.24 \end{array} \qquad \begin{array}{r} 6.23 \\ 8.01 \\ \hline 14.24 \end{array} \qquad \begin{array}{r} 3.56 \\ 4.45 \\ \hline 8.01 \end{array} \qquad \begin{array}{r} 6.23 \\ 11.57 \\ \hline 17.80 \end{array} \qquad \begin{array}{r} 7.12 \\ 8.01 \\ \hline 15.13 \end{array}$$

The owner was horrified and said, "Why are you doing all this work? Didn't I tell you that each and every book is selling for 89 cents during the sale?"

[Before reading any further, try to find a way by which Walt could use the information the owner gave him to simplify his job. Remember, the saleslips which are brought to Walt do not tell the number of books purchased; they tell only the total cost in each department.]

Emphasize that the table is a device for obtaining the sum of certain numbers by adding other numbers. For example, the student must recognize that the problem of finding the sum of, say, 534 and 979, is solved by finding the "correspondents" of 534 and 979, adding the correspondents, and then finding the correspondent of that sum. Do not encumber the process by mentioning books and cents; merely add numbers.

* * *

The way of constructing this table and others in the section is not important to the development. Naturally, students will be interested in how to make such tables, but this is incidental to the main idea of isomorphism.

* * *

Walt's table is an illustration of an isomorphism with respect to addition.

* * *

In Exercises 8 and 9 the students will have to include 0 in the left-hand list and its correspondent in the right-hand list.

The store closed during the noon hour and the owner showed Walt a shortcut. He made a table like the one shown at the right of this page.

89 . . .	1
178 . . .	2
267 . . .	3
356 . . .	4
445 . . .	5
534 . . .	6
623 . . .	7
712 . . .	8
801 . . .	9
890 . . .	10
979 . . .	11
1068 . . .	12
1157 . . .	13
1246 . . .	14
1335 . . .	15
1424 . . .	16
1513 . . .	17
1602 . . .	18
1691 . . .	19
1780 . . .	20

He told Walt that with this table he could do every one of his additions mentally by using the handwritten numerals instead of the typed numerals. Add the "easy" numbers instead of the "hard" numbers. Add 534 and 979 by the following scheme:

$$\begin{array}{rcr} 534 & \longrightarrow & 6 \\ \underline{979} & \longrightarrow & \underline{11} \\ 1513 & \longleftarrow & 17 \end{array}$$

EXERCISES

A. Use Walt's table to add mentally the following pairs of numbers.

1. 267 356	2. 445 890	3. 623 623
4. 712 1068	5. 178 1513	6. 356 89
7. 1246 356	8. 0 1513	9. 1691 0

Part B contains another illustration of an isomorphism with respect to addition. Note that we use handscript for the numerals in one column and typescript for the numerals in the other column. We use this device to emphasize that we are dealing with ordered pairs of numbers. The two scripts help us distinguish first components from second components.

* * *

The table is to be completed in Part C in such a way that the isomorphism is maintained.

* * *

Part D points out that the essence of the isomorphism is not the shortcut notion which characterized the preceding tables. The <u>shortcut idea was merely motivation</u>. The exercises in Part D require the student to use the table "both ways".

* * *

There is some danger that the students will think that the tables in Part D and E are tables giving "common fraction-decimal fraction equivalents". You should take the necessary steps to correct such notions at the outset of the work in Part D.

B. The table at the right can be used to add the large numbers in the same manner as you used the table in Part A. Add mentally the following pairs of numbers.

1,437,345	 5
1,724,814	 6
2,012,283	 7
2,299,752	 8
2,587,221	 9
2,874,690	 10
3,162,159	 11
3,449,628	 12
3,737,097	 13
4,024,566	 14
4,312,035	 15
4,599,504	 16
4,886,973	 17
5,174,442	 18

1. 2,874,690
   $\underline{2,299,752}$

2. 1,724,814
   $\underline{3,162,159}$

3. 2,012,283
   $\underline{2,012,283}$

4. 3,449,628
   $\underline{1,437,345}$

C. Complete the top of the table in Part B. That is, find the numbers corresponding to the numbers 4, 3, 2, and 1.

D. The table at the right does not give you a shortcut in adding numbers. However, you can use the same pattern to add certain numbers as you did in the "shortcut" tables. For example, the names of the numbers in Exercise 1 on the next page appear in the right-hand column. Instead of adding these numbers directly, add the corresponding numbers from the left-hand column. In Exercise 2 you would reverse the procedure. Check a few of the exercises to convince yourself that the table works.

.030861	... $+36\frac{3}{4}$
.013716	... $+16\frac{1}{3}$
.041148	... +49
.020574	... $+24\frac{1}{2}$
.006858	... $+8\frac{1}{6}$
.024003	... $+28\frac{7}{12}$
.037719	... $+44\frac{11}{12}$
.034290	... $+40\frac{5}{6}$
.027432	... $+32\frac{2}{3}$
.068580	... $+81\frac{2}{3}$

The table in Part E shows an isomorphism with respect to multiplication. So does the table in Part F on the next page. Continue to emphasize the isomorphism idea and leave out any reference to shortcuts. These tables are of no importance in computation.

* * *

Another table which illustrates an isomorphism with respect to multiplication is a table which lists ordered pairs from the square function.

1	2	3	4	5	6	7	8	9	10	11	12
1	4	9	16	25	36	49	64	81	100	121	144

$$x_1 \longleftrightarrow (x_1)^2$$

$$\underline{x_2 \longleftrightarrow (x_2)^2}$$

$$x_1 x_2 \longleftrightarrow (x_1 x_2)^2 = (x_1)^2 (x_2)^2$$

1. $+8\frac{1}{6}$ / $\underline{+16\frac{1}{3}}$

2. .013716 / $\underline{.020574}$

3. $+16\frac{1}{3}$ / $\underline{+24\frac{1}{2}}$

4. .020574 / $\underline{.020574}$

5. $+40\frac{5}{6}$ / $\underline{+40\frac{5}{6}}$

6. .030861 / $\underline{.037719}$

E. The tables in the preceding exercises were used for addition problems. The table at the right can be used in multiplication problems!

.5	...	+2
.25	...	+4
.2	...	+5
.125	...	+8
.1	...	+10
.0625	...	+16
.05	...	+20
.04	...	+25
.03125	...	+32
.025	...	+40

Sample. What is the product of .05 and .5?

$$\begin{array}{rcr} .05 & \longrightarrow & 20 \\ \times\ \underline{.5} & \longrightarrow & \times\ \underline{2} \\ .025 & \longleftarrow & 40 \end{array}$$

Find the products of the pairs of numbers by using the corresponding numerals in the other column.

1. .5 / $\underline{\times\ .2}$

2. +8 / $\underline{\times\ +5}$

3. .0625 / $\underline{\times\ .5}$

4. +8 / $\underline{\times\ +4}$

5. .125 / $\underline{\times\ .25}$

6. .2 / $\underline{\times\ .2}$

Parts F and G are at different levels of abstraction. Note that in the table in F, the numerals are not "in order". In the table in G, the table itself has been "taken apart". Let the student discover that the key to using the table in G is "handscript vs. typescript".

* * *

An important idea in isomorphism is one-to-one correspondence; the tabular arrangement is merely a convenient but not a necessary device to indicate this correspondence. Part G is an attempt to give the student an intuitive feeling for this idea. Here you must use your own judgment regarding how much philosophical discussion is wise. You might, for example, ask students what properties of a table are absolutely essential. Suppose all the numerals were in typescript in Part G ? Or, suppose the numerals were merely scattered over the page with no indication of pairs, or ordering of the elements within the pairs?

* * *

For your own interest, consider a table of logarithms. Here is an example of an isomorphism between two sets with respect to addition in one set and multiplication in the other:

2	...	0.30103
3	...	0.47712
6	...	0.77815

$$\begin{array}{rcr} 3 & \longrightarrow & 0.47712 \\ \times 2 & \longrightarrow & +0.30103 \\ \hline 6 & \longleftarrow & 0.77815 \end{array}$$

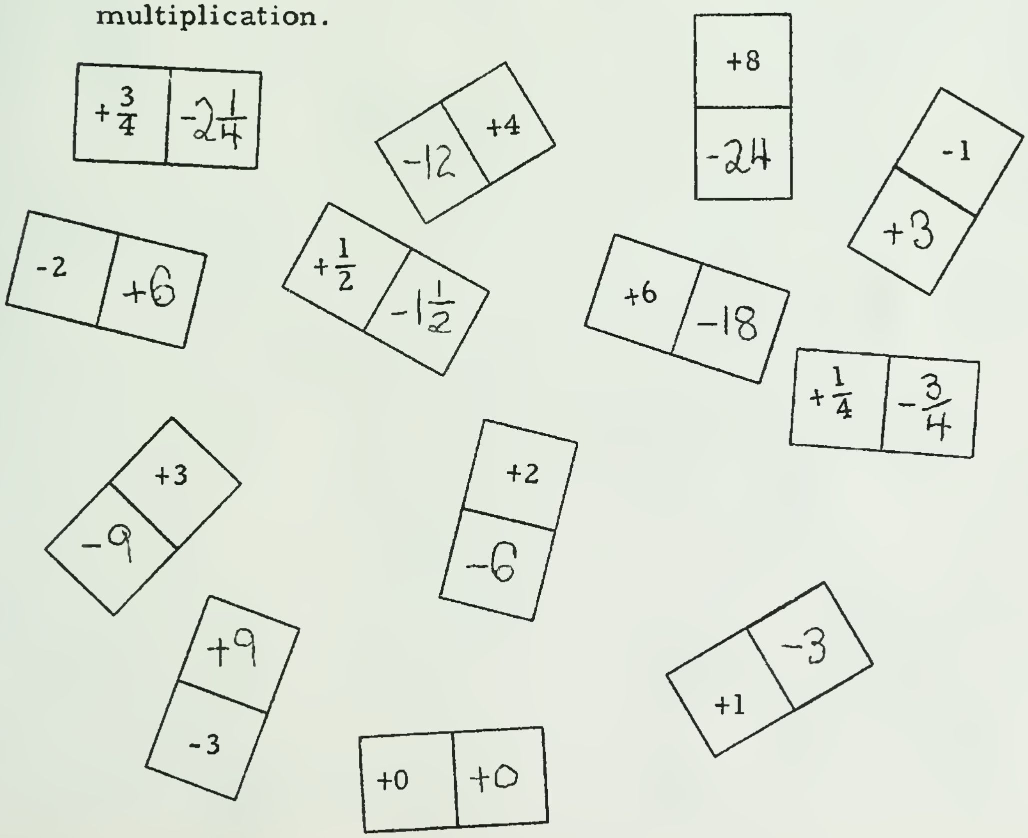
F. Below is a table which you can use for multiplication just as you used the table in Part E.

$+2$	$+5$	$+\frac{1}{3}$	$+\frac{1}{5}$	$+\frac{1}{9}$	$+\frac{2}{3}$	$+1$	$+\frac{5}{9}$	$+\frac{10}{27}$	$+\frac{20}{81}$
$\frac{1}{2}$	$\frac{1}{5}$	3	5	9	$1\frac{1}{2}$	1	$1\frac{4}{5}$	2.7	4.05

Find the products of the pairs of numbers in each exercise. Use the table even though it doesn't save you any work.

1. $3 \times \frac{1}{2}$
2. $(+\frac{1}{3}) \times (+2)$
3. $9 \times \frac{1}{5}$
4. $(+\frac{1}{9}) \times (+5)$
5. $1\frac{1}{2} \times 2.7$
6. $(+\frac{2}{3}) \times (+\frac{10}{27})$

G. Below is a "table" which can be used just as you used the other tables in this section. Figure out how to use the table and see whether it works for addition or for multiplication.



Before starting Part H, ask your students the following question:

> Do you think there is a table which "works" for
> <u>both</u> addition and multiplication?

Leave the question open.

* * *

Students had some difficulty in following the instructions for Part H. We suggest that you develop the Sample step by step at the blackboard instead of trying to have the students read it directly from the text.

* * *

In the solution to the Sample, some students objected to the diagram:

$$
\begin{array}{lccccc}
\text{(A)} & 4 & \times & 2 & \overset{?}{=} & 4 \\
 & \downarrow & & \downarrow & & \uparrow \\
\text{(B)} & 2 & \times & 1 & = & 2 \\
 & & & \multicolumn{3}{c}{\xrightarrow{\qquad\qquad}}
\end{array}
$$

because we had already selected 4, 2, and 8 from row A. These students preferred instead the diagram:

$$
\begin{array}{lccccc}
\text{(A)} & 4 & \times & 2 & = & 8 \\
 & \downarrow & & \downarrow & & \downarrow \\
\text{(B)} & 2 & \times & 1 & \overset{?}{=} & 4
\end{array}
$$

Students should realize that either of these schemes is correct; each shows that the table does not work for multiplication.

The discussion of these two schemes for checking applies to all the exercises in this section in which students are checking tables.

* * *

Exercise 7 of Part H culminates in the idea toward which we have been driving--the isomorphism between the positive directed numbers and the numbers of arithmetic.

H. You have seen that some tables work for addition and some for multiplication. Below you are given 8 rows. Any two of the rows considered together give you a table. The table may work for multiplication, for addition, or for neither of these or for both of them.

A	6	16	4	12	2	10	8	14
B	+3	+8	+2	+6	+1	+5	+4	+7
C	$-\frac{1}{3}$	$-\frac{1}{8}$	$-\frac{1}{2}$	$-\frac{1}{6}$	-1	$-\frac{1}{5}$	$-\frac{1}{4}$	$-\frac{1}{7}$
D	$+\frac{2}{3}$	$+1\frac{7}{9}$	$+\frac{4}{9}$	$+1\frac{1}{3}$	$+\frac{2}{9}$	$+1\frac{1}{9}$	$+\frac{8}{9}$	$+1\frac{5}{9}$
E	4	9	3	7	2	6	5	8
F	-9	-24	-6	-18	-3	-15	-12	-21
G	3	8	2	6	1	5	4	7
H	$\frac{1}{3}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{6}$	1	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{7}$

Sample. Consider the table with the rows A and G. Check the table for multiplication and for addition.

A	6	16	4	12	2	10	8	14
G	3	8	2	6	1	5	4	7

Solution. Pick three numbers whose names appear in, say, row A, such that the product of two of the numbers is the third. Thus, try 4, 2, and 8.

$$\begin{array}{lccccc} (A) & 4 & \times & 2 & \overset{?}{=} & 4 \\ & \downarrow & & \downarrow & & \uparrow \\ (G) & 2 & \times & 1 & = & 2 \\ & & & & \longrightarrow & \end{array}$$

We think that in Exercise 7 students will be quite innocently drawn into actually carrying out the "correspondence procedure" they used in the other exercises. We think they will regard the positive numbers as things which are distinct from the numbers of arithmetic, but that both sets "act" alike.

* * *

In Exercise 7 the students have answered the question about whether there is a table which works for both multiplication and addition.

Note that the '2' in row G corresponds to the '4' in row A. But, 4 is not the product of 4 and 2. So, the table does not work for multiplication. (Why should you be convinced that it does not work by this single case?)
Check the table for addition. Pick three numbers whose names appear in, say, row A such that the sum of two of the numbers is the third. Try 4, 8, and 12.

$$\begin{array}{lccccccc} (A) & 4 & + & 8 & \stackrel{?}{=} & 12 \\ & \downarrow & & \downarrow & & \uparrow \\ (G) & 2 & + & 4 & = & 6 \\ & & & \longrightarrow & & \end{array}$$

Since 4 + 8 is 12, then the table seems to work for addition. Try other numbers.

In each of the following exercises test the tables with the given rows.

1. G and H
2. F and G
3. A and D
4. A and E
5. B and F
6. B and H
7. B and G

1.08 Positive numbers and the numbers of arithmetic. -- In adding or multiplying pairs of directed numbers in earlier sections you probably found that it was easiest when both numbers were positive. Let's see why working with positive numbers is easy.

Find the sum of, say, +7 and +2. Think carefully of what you do as you find the sum. Aren't you really using a table that looks like this?

Positive numbers	+7	+2	+9
Numbers of arithmetic	7	2	9

If by any chance a student should demand an explanation of the difference between, say 6 and +6, your only answer is to return to interpretations of '6' and '+6'. The important thing at this time is that the student does not acquire the incorrect point of view that the numbers of arithmetic and the positive directed numbers are identical. Certainly, we do not want to say that a directed number is just a number of arithmetic with plus sign tacked on!

* * *

Part A indicates the process by which most of us add negative numbers. Notice how thinking about a table of this type eliminates the need for the statement of a rule.

Do not omit Part B which appears at the top of page 1-39.

Don't you actually think that

$$7 + 2 = 9,$$

and therefore,

$$+7 + (+2) = +9 ?$$

So, what you do when you find the sum of +7 and +2 is exactly what you did when you used the tables in the preceding section.

$$\begin{array}{ccccc} +7 & + & (+2) & = & +9 \\ \downarrow & & \downarrow & & \uparrow \\ 7 & + & 2 & = & 9 \end{array}$$

And, aren't you using a table like the following when you multiply two positive numbers, say, +4 and +6?

Positive numbers	+4	+6	+24
Numbers of arithmetic	4	6	24

Thus, for multiplying and adding positive numbers you have a kind of mental picture of a large table. One row of the table contains names of all the positive numbers and the other contains names of all the numbers of arithmetic. This table works for both addition and multiplication. (You checked part of this table in Exercise 7, Part H of the preceding exercises.) Later in the unit you will find that the table also works for subtraction and division of positive numbers.

EXERCISES

A. Check to see that the following table works for addition. (This may help you remember how to add negative numbers.)

-1	$-1\frac{1}{2}$	-2	-3	$-1\frac{2}{3}$	$-3\frac{1}{6}$	$-3\frac{1}{2}$	$-3\frac{2}{3}$	$-4\frac{1}{6}$	$-5\frac{1}{2}$	-6
1	$1\frac{1}{2}$	2	3	$1\frac{2}{3}$	$3\frac{1}{6}$	$3\frac{1}{2}$	$3\frac{2}{3}$	$4\frac{1}{6}$	$5\frac{1}{2}$	6

6. '0' and '10 + (-10)' are names of the same number.

7. The opposite of $\frac{7}{3}$ is $-\frac{3}{7}$.

8. Negative numbers act like arithmetic numbers with respect to multiplication. For example:

$$7 \times 7 = 49$$
$$(-7) \times (-7) = 49$$

9. I can make +92 an arithmetic number if I just cross out ' + '.

10. The reciprocal of $-\frac{5}{3}$ is $\frac{3}{5}$.

[Your students may not be familiar with the term 'reciprocal'. If they aren't you should omit Exercises 4 and 10 of this set.]

Be sure that students understand that when someone claims that the equation:

$$+6 = 6$$

is true, he intends that the '6' in the right-hand member is an abbreviation for '+6'. He does not mean (or, if he does, he is wrong from our point of view) that the positive real number 6 is the same thing as the unsigned real number 6.

Call their attention to the fact that '+0', '-0', and '0' are all names for the number 0; and that we shall henceforth write '0' in lieu of the other names. Also, now is the time to state clearly that among the directed numbers there are three kinds of numbers:

positive numbers
negative numbers
zero

The directed number 0 is neither positive nor negative.

Supplementary exercises:

Tell whether it is correct to do or say each of the following.

1. Write a ' - ' in front of the number 5 and make it negative 5.
2. We agreed to write '+63' for '63'; ' $+\frac{7}{12}$ ' for $\frac{7}{12}$.
3. I know now that I had been working with positive numbers all the time when I was in grade school.
4. The reciprocal of $\frac{+13}{8}$ is $\frac{8}{13}$.
5. If it is correct to write '(+3) + 25 = 3 + 25', then it is correct to write '(+3) × (+25) = 3 × 25'.

(continued on T. C. 39C)

Admittedly, after stressing the difference between the numbers of arithmetic and the positive directed numbers, it is a little strange to "confuse" their names. But since it is so widely practiced we felt it was necessary here to conform. In the boxed statement, we are saying in effect, that when you look at the symbol, say '3', you cannot tell (except, at times, from the context) whether the symbol is intended to name a number of arithmetic or a positive number.

* * *

Teachers report that students have considerable difficulty in getting accustomed to writing '6' instead of '+6', '0' instead of '+0', etc. No doubt it will be necessary for you to take some time in class to discuss the fact that we are adopting a convention that is widely used--i.e., writing a shorter name for something--and explain that this is quite common, though not technically correct. In the case here, the reason for <u>wanting</u> to write a shorter name is to save time and space [and to "get along" with the rest of the world where most people regard the positive real numbers as identical with the unsigned reals]; the reason it is <u>permissible</u> is that the confusion of names will not cause computational errors. The isomorphism between the two sets of numbers permits you to confuse the names without worry.

'$6 + 2 = 8$' can mean $(+6) + (+2) = (+8)$.

'$5 \times 3 = 15$' can mean $(+5) \times (+3) = (+15)$.

Now if we write '$6 + (-2) = 4$', the reader knows that we <u>must</u> mean $(+6) + (-2) = +4$. In other words, if one numeral in an expression names a directed number, then the reader must understand that all numbers named in the expression are directed.

(continued on T. C. 39B

B. Does the table in Part A work for multiplication?

OTHER NAMES FOR POSITIVE NUMBERS

You are familiar with the use of several different names for the same number. For example, '+8', '+4 + (+4)', and '+2 × (+4)' are names for the same number. You have been told that the numbers of arithmetic add, multiply, subtract, and divide in a way exactly corresponding to that of the positive numbers. Therefore, it will cause no trouble if we use the names of arithmetic numbers as names for the positive numbers. Thus, we shall write '8' instead of '+8', or '$3\frac{1}{2}$' instead of '$+3\frac{1}{2}$' whenever we want to.

> Throughout this book we shall write
>
> '0' for '+0' (or '-0')
>
> '1' for '+1'
>
> '2' for '+2'
>
> '$5\frac{1}{5}$' for '$+5\frac{1}{5}$'
>
> '153' for '+153'
>
> [and so on for all positive numbers]
>
> whenever it is convenient.

EXERCISES

A. Add the directed numbers listed in each of the following exercises. (Notice the two ways of naming positive numbers.)

1. 6, -2
2. +5, 0.01
3. 48, 3.02
4. -9, $3\frac{1}{2}$
5. $-\frac{1}{7}$, $3\frac{1}{3}$
6. .01, $-\frac{1}{2}$

B. Multiply the directed numbers listed in each of the following exercises.

1. 6, -3
2. $7\frac{1}{2}$, $+\frac{1}{3}$
3. $-\frac{1}{6}$, $-\frac{2}{4}$
4. 5, $\frac{1}{4}$
5. -4, $3\frac{1}{3}$
6. $-\frac{17}{4}$, $-\frac{4}{17}$

If your students question the need for symbols of grouping, show them an example such as this:

What is a simpler name for the number named by:

$$7 \times 3 + (^{-}2)?$$

To show the advantage in having more than one <u>kind</u> of grouping symbol, let the class compare these two expressions:

a) $(((3 \times 2) + (4 \times 1)) + 5) \times 1$

b) $\{[(3 \times 2) + (4 \times 1)] + 5\} \times 1$

1.09 Using symbols of groupings.--Before we continue with our study of directed numbers we need to learn more about expressions for numbers.

Consider the two expressions

'(7 + 5) + 3' and '8 + (9 + 2)'

The parentheses in '(7 + 5) + 3' indicate that the expression stands for the sum of 7 + 5 and 3. The parentheses in '8 + (9 + 2)' indicate that the expression stands for the sum of 8 and 9 + 2. Thus, we can write:

'(7 + 5) + 3' for '12 + 3'

and

'8 + (9 + 2)' for '8 + 11'

Parentheses, '(' and ')', are grouping symbols. They serve a purpose in mathematical expressions similar to the purpose punctuation marks serve in English expressions--they help you get the meaning the writer intended when he wrote an expression. There are other symbols for grouping besides parentheses. Two examples of such symbols are brackets, '[' and ']', and braces, '{' and '}'. Watch how these grouping symbols are used in the following examples.

Example 1. Does the expression:

[5 + (-3)] + (+6)

stand for the same number as the expression:

5 + [-3 + (+6)]

Solution. Note that in these expressions we use '(' and ')' around the '+6' to avoid confusing the addition sign with the '+' in '+6'. We use '[' and ']' as grouping symbols because parentheses have already been used and two sets of parentheses would be confusing. We know that we can write:

'[5 + (-3)] + (+6)' for '2 + (+6)'

and that

2 + (+6) = +8

Also, we know that we can write:

'5 + [-3 + (+6)]' for '5 + (+3)'

and that

5 + (+3) = +8

So, the two expressions are numerals for the same number, +8.

<u>Example 2</u>. Give a simpler name for the number expressed by:

$$\{[(2 \times 3) + 4] + 5\} \times 6$$

<u>Solution</u>. We know that the given expression is a name for a number. The given expression, however, is quite complicated in appearance. We are seeking another name or expression which is simpler looking but still stands for the same number.

We want a simpler name for

Then we need to find a simpler name for

But then we need to find a simpler name for

which we can do by finding a simpler name for

$$\{[(2 \times 3) + 4] + 5\} \times 6$$

$$[(2 \times 3) + 4] + 5$$

$$(2 \times 3) + 4$$

$$(2 \times 3)$$

We know a simpler name for 2×3. So, we write the following:

$$\begin{aligned} &\{[(2 \times 3) + 4] + 5\} \times 6 \\ = &\{[\quad 6 \quad + 4] + 5\} \times 6 \\ = &\{\quad 10 \quad + 5\} \times 6 \\ = &\quad 15 \quad \times 6 \\ = &\quad 90 \end{aligned}$$

Most people would agree that ' 90 ' is a simpler name for 90 than '$\{[(2 \times 3) + 4] + 5\} \times 6$'.

EXERCISES

Write the simplest name you can find for the number named by each of the following expressions.

1. $2 + (-8)$
2. $3 + (-9)$
3. $(5 + 2) + 1$
4. $6 + (8 + 7)$
5. $(9 \times 2) \times 3$
6. $4 \times (7 \times 3)$

The principles of arithmetic are very basic in mathematics. With a good understanding of these principles, the job of learning algebra becomes relatively easy. We shall make constant reference to these principles in later units. The students ought to become accustomed to the names for the principles and to be able to illustrate them. [The verbalizations of the principles do not need to be considered until Unit 2.] You can help students remember the names for the principles by giving them mnemonic devices such as:

commutative ..... commuting back and forth between home and work.

associative ...... in the expression '2 + 3 + 4', the '3' can associate with either the '2' or the '4'.

distributive ...... in the expression '5 × (2 + 3)' the ''multiplication by 5'' is distributed or spread over the '2' and the '3'.

* * *

Although it is not necessary to make an issue of it at this time, you should mention the fact that students have assumed that these principles hold for the numbers of arithmetic. The principles can be proved [see Thurston, op. cit.] from other, more basic, assumptions. However, it will seem a bit strange to students to even question these principles because they have lived with the principles for at least eight years.

7. $[-3 + (-2)] + (-8)$
8. $6 + [(-8) + 5]$
9. $[-3 \times (-2)] \times (-7)$
10. $-3 \times [-2 \times (-7)]$
11. $(5 \times 4) + 3$
12. $6 + (2 \times 7)$
13. $[(-5) \times 4] + (-2)$
14. $-4 + [(-3) \times (-2)]$
15. $3 \times (2 + 7)$
16. $4 \times (6 + 1)$
17. $(3 \times 2) + (3 \times 7)$
18. $(4 \times 6) + (4 \times 1)$
19. $-3 \times [(-2) + (-7)]$
20. $5 \times [(-3) + 8]$
21. $[-3 \times (-2)] + [-3 \times (-7)]$
22. $[5 \times (-3)] + (5 \times 8)$
23. $2 \times \{[4 \times (3 + 2)] + [4 \times (5 + 9)]\}$
24. $2 \times \{[(4 \times 3) + (4 \times 2)] + [(4 \times 5) + (4 \times 9)]\}$

1.10 Directed numbers and the principles of arithmetic.-- The numbers of arithmetic have certain properties which you made use of time and again in your earlier work in mathematics. For example, you know that if you select a number, say, 5 and multiply it by 1, the product is 5. If you select $17\frac{1}{4}$ and multiply it by 1 the product is $17\frac{1}{4}$. Pick any number at all and multiply it by 1. What number do you get as the product? These examples illustrate an important principle which we shall call the principle of 1.

Another important property of the numbers of arithmetic is shown in the following example. Pick a number, say, 8. Pick another number, say, 4. Now, find the sum of these numbers. Of course, the sum is 12. But, there are two ways of getting the sum. You can take 8 first and add 4 to 8. Or, you can take 4 first and add 8 to 4. In each case you get the same sum 12. Hence, we can say that '$8 + 4$' and '$4 + 8$' are names for the same number. This is an example of another important principle, the commutative principle for addition. There are many other principles which you have used in arithmetic. We shall illustrate the most important ones a little later.

Now, it would be very convenient if these principles could be applied to directed numbers. For example, if the commutative principle worked for the addition of directed numbers, we could say, for example, that '$+2 + (-7)$' and '$-7 + (+2)$' were names for the same directed number. Then, in finding the sum

The associative principle tells you, for example, that

$$'[(4 + 9) + 3] + 7' \quad \text{and} \quad '4 + [9 + (3 + 7)]'$$

stand for the same number. Therefore, we agree that $4 + 9 + 3 + 7$ is the same number as $[(4 + 9) + 3] + 7$ or $4 + [9 + (3 + 7)]$, and we agree to omit the parentheses.

Similarly, since all of the following expressions stand for the same number:

$$(5 \times 2) \times (9 \times 3),$$

$$[(5 \times 2) \times 9] \times 3,$$

$$5 \times [2 \times (9 \times 3)],$$

we agree to omit the parentheses, and simply write:

$$5 \times 2 \times 9 \times 3$$

when we mean $[(5 \times 2) \times 9] \times 3$.

* * *

After discussing the associative principle for addition and multiplication, someone [perhaps, you] may ask whether there is an associative principle for division. You might have the students consider these:

$$(0 \div 1) \div 1 \overset{?}{=} 0 \div (1 \div 1)$$

$$(0 \div 2) \div 3 \overset{?}{=} 0 \div (2 \div 3)$$

Then ask whether someone can suggest an expression showing that associativity for division does not "work".

Here is one you might use:

$$(6 \div 3) \div 2 \overset{?}{=} 6 \div (3 \div 2)$$

$$1 \neq 4$$

In similar fashion, you can consider commutative principles for subtraction and division, or the associative principle for subtraction.

[Incidentally, do not avoid the use of the handy word 'cancel'. Keep in mind that cancelling is something one does to symbols but not to numbers. To cancel a symbol, you just draw a short diagonal slash through it, or erase it. You can't cancel numbers. For example, you may multiply the numerator-number and the denominator-number of a fraction by some number such as $\frac{1}{3}$ or $\frac{1}{2}$. And you show the result of this operation by cancelling numerator and denominator and writing other numerals close to the cancelled ones.]

$$
\begin{aligned}
\text{(IV)}\quad 9\tfrac{1}{2} \times 4\tfrac{1}{3} &= (9 + \tfrac{1}{2}) \times (4 + \tfrac{1}{3}) \\
&= [(9 + \tfrac{1}{2}) \times 4] + [(9 + \tfrac{1}{2}) \times \tfrac{1}{3}] \\
&= [4 \times (9 + \tfrac{1}{2})] + [\tfrac{1}{3} \times (9 + \tfrac{1}{2})] \\
&= [36 + 2] + [3 + \tfrac{1}{6}] \\
&= 41\tfrac{1}{6}
\end{aligned}
$$

* * *

We have not included a discussion of continued sums and continued products. However, some teachers reported misunderstanding of these on the part of the students. You may want to ask students what symbols like

'4 + 9 + 3 + 7' and '5 × 2 × 9 × 3'

stand for.

(continued on T. C. 43D)

Point out to the students that multiplication distributes over subtraction as well as over addition. Then give a shortcut such as:

(II) $5 \times 9997 = ?$

$$
\begin{aligned}
5 \times 9997 &= 5 \times (10000 - 3) \\
&= (5 \times 10000) - (5 \times 3) \\
&= 50000 - 15 \\
&= 49985
\end{aligned}
$$

(III) What is the mathematical basis for the ''cancelling'' kids do in a problem such as:

$$\frac{\overset{1}{\cancel{4}}}{\underset{5}{\cancel{15}}} \times \frac{\overset{1}{\cancel{3}}}{\underset{2}{\cancel{8}}} = \frac{1}{10}$$

$$\frac{4}{15} \times \frac{3}{8} = \overbrace{\frac{4 \times 3}{15 \times 8} = \frac{3 \times 4}{15 \times 8}}^{\text{commutativity}} = \frac{\overset{1}{\cancel{3}}}{\underset{5}{\cancel{15}}} \times \frac{\overset{1}{\cancel{4}}}{\underset{2}{\cancel{8}}} = \frac{1}{10}$$

(continued on T. C. 43C)

Note that in checking to see that the commutative principle for addition holds for directed numbers, the procedure is to find simpler expressions to replace the expressions on each side of the equal sign. In other words, we know that

$$+9 + (+8) = +8 + (+9)$$

because we know that

$$+9 + (+8) = +17 \quad \text{and} \quad +8 + (+9) = +17$$

and not because we are assuming that the commutative principle for addition holds for directed numbers.

* * *

In illustrating the principles with the numbers of arithmetic (pages 43-45), have the students check as above. There is opportunity here for maintenance and remedial work in computation. There is also opportunity to demonstrate some computational shortcuts as illustrated in Part C on page 46. You may find that merely presenting one principle after the other becomes somewhat tedious for the students. So, intersperse the presentation of the principles with interesting shortcuts and explanations of some of the algorisms the students learned in grade school. Encourage mental manipulations wherever possible.

(I) $\begin{array}{r} 41 \\ \times\ 3 \\ \hline ? \end{array}$

$$3 \times 41 = \overbrace{3 \times (40 + 1) = (3 \times 40) + (3 \times 1)}^{\text{distributive principle}}$$
$$= 120 + 3$$
$$= 123$$

(continued on T. C. 43B)

of two directed numbers, it would make no difference which of the numbers you considered first. Let us see if directed numbers have properties which satisfy the commutative principle for addition.

First, we select a pair of directed numbers, say, +2 and -7. Then, we compute the sum obtained by adding -7 to +2.

$$+2 + (-7) = -5$$

Next, we compute the sum obtained by adding +2 to -7.

$$-7 + (+2) = -5$$

In both cases the sum is -5. So, the commutative principle for addition holds, at least, for these directed numbers. We could try to convince ourselves that it holds in general by using other pairs of directed numbers:

$$+9 + (+8) = +17 \;\ldots\ldots\; +8 + (+9) = +17$$
$$-4 + (-3) = -7 \;\ldots\ldots\; -3 + (-4) = -7$$
$$-5 + (+8) = +3 \;\ldots\ldots\; +8 + (-5) = +3$$
$$+4 + (-2) = +2 \;\ldots\ldots\; -2 + (+4) = +2$$
$$-6 + 0 = -6 \;\ldots\ldots\; 0 + (-6) = -6$$
$$+\frac{1}{2} + (-\frac{1}{3}) = +\frac{1}{6} \;\ldots\ldots\; -\frac{1}{3} + (+\frac{1}{2}) = +\frac{1}{6}$$

etc.

PRINCIPLES OF ARITHMETIC

We now illustrate certain important basic principles which hold for the numbers of arithmetic.

The Principle of 1

$$6 \times 1 = 6$$
$$5 \times 1 = 5$$
$$\frac{3}{4} \times 1 = \frac{3}{4}$$

etc.

Miss Blair's students suggested that the principle of 0 be stated as two principles, namely a <u>principal of 0 for addition</u> and a <u>principle of 0 for multiplication</u>. This is an excellent suggestion which we shall follow in the next revision. If you use it with your students this year, you will need to make a few corresponding changes later on in Unit 1 and in Unit 2.

The Principle of 0

$$4 + 0 = 4$$

$$5\frac{1}{2} + 0 = 5\frac{1}{2}$$

$$\frac{23}{18} + 0 = \frac{23}{18}$$

etc.

and

$$4 \times 0 = 0$$

$$5\frac{1}{2} \times 0 = 0$$

$$\frac{23}{18} \times 0 = 0$$

etc.

The Commutative Principle for Addition

$$9 + 3 = 3 + 9$$

$$7 + \frac{3}{4} = \frac{3}{4} + 7$$

$$3\frac{1}{3} + 128.5 = 128.5 + 3\frac{1}{3}$$

etc.

The Commutative Principle for Multiplication

$$4 \times 2 = 2 \times 4$$

$$3 \times \frac{1}{2} = \frac{1}{2} \times 3$$

$$0.016 \times 4.32 = 4.32 \times 0.016$$

etc.

The Associative Principle for Addition

$$1 + (2 + 3) = (1 + 2) + 3$$

$$(4 + \frac{1}{2}) \div 19 = 4 + (\frac{1}{2} + 19)$$

$$(7 + 0.2) + \frac{4}{5} = 7 + (0.2 + \frac{4}{5})$$

etc.

8) Add the opposite of the reciprocal of one of them to the other to get 0.

9) If one of them is a negative number then the other is a positive number.

10) The reciprocal of the opposite of one number is the opposite of the other number.

$$+6 + (-6) = 0,$$

$$-7 + 7 = 0,$$

$$\frac{1}{4} + (-\frac{1}{4}) = 0,$$

etc.

Students should be able to see with little difficulty the parallelism between these two principles. They should note that every number has exactly one opposite and that every number, except 0, has exactly one reciprocal.

To help fix the ideas of the principle of opposites and the principle of reciprocals, here are

Supplementary exercises:

If two numbers are reciprocals, which of the following statements about them are true? Which are false? Give a counter-example for each false statement.

1) Multiply one of them by -1 to get the other.
2) Add the opposite of one number to the other to get 1.
3) The opposites of the two numbers are reciprocals.
4) Add the reciprocal of the first number to the second number, and you get twice the second number.
5) Divide 1 by one of the numbers to get the other.
6) The quotient of one of them by the other is the reciprocal of their product.
7) The opposite of their product is -1.

(continued on T. C. 45C)

The organization of steps in Part B is somewhat difficult to follow as it is now presented. There are expressions for three numbers (Exercises 1, 2, and 3) and the student is to identify each step in the simplification of each of the expressions. For example, the first expression to be considered is '$5 \times (9 \times \frac{1}{5})$'. In Exercise 1(a), the commutative principle for multiplication is applied to give the expression '$5 \times (\frac{1}{5} \times 9)$'. The associative principle for multiplication is applied in Exercise 1 (b).

To explain Exercise 1 (c) you should introduce here another principle called <u>the principle of reciprocals.</u> Then the statement '$5 \times \frac{1}{5} = 1$' is an instance of that principle. Other instances of the principle of reciprocals are:

$$\frac{1}{8} \times 8 = 1,$$

$$\frac{2}{3} \times \frac{3}{2} = 1,$$

$$-\frac{1}{3} \times (-3) = 1.$$

The principle of reciprocals is mentioned on page 1-56, but some of our teachers have suggested that it is helpful to introduce it here.

As a parallel to the principle of reciprocals, you should also introduce the <u>principle of opposites,</u> instances of which are

(continued on T. C. 45B)

<u>The Associative Principle for Multiplication</u>

$$4 \times (7 \times 8) = (4 \times 7) \times 8$$

$$3 \times (\frac{1}{3} \times 7) = (3 \times \frac{1}{3}) \times 7$$

$$(0.051 \times .03) \times 10 = 0.051 \times (.03 \times 10)$$

etc.

<u>The Distributive Principle</u>

$$5 \times (2 + 4) = (5 \times 2) + (5 \times 4)$$

$$3 \times (\frac{1}{2} + 2) = (3 \times \frac{1}{2}) + (3 \times 2)$$

$$\frac{5}{8} \times (\frac{1}{7} + \frac{3}{11}) = (\frac{5}{8} \times \frac{1}{7}) + (\frac{5}{8} \times \frac{3}{11})$$

etc.

EXERCISES

<u>A</u>. Read each of the following statements and tell the principle it illustrates.

1. $7 + 0 = 7$
2. $3 \times 1 = 3$
3. $4 + 1 = 1 + 4$
4. $5 \times 8 = 8 \times 5$
5. $0 \times 9 = 9 \times 0$
6. $1 \times \frac{1}{2} = \frac{1}{2} \times 1$
7. $8 \times 0 = 0$
8. $6 + 0 = 0 + 6$
9. $(3 + 1) + 7 = 3 + (1 + 7)$
10. $(7 \times 9) \times 0 = 7 \times (9 \times 0)$
11. $(5 + 8) \times 4 = 4 \times (5 + 8)$
12. $6 \times (9 + 2) = (6 \times 9) + (6 \times 2)$

<u>B</u>. You use the principles of arithmetic when you find a simpler name for a number. In the following exercises, three simplifications are carried out step by step. In each exercise state which principle is being used.

1. Simplify: $5 \times (9 \times \frac{1}{5})$

(a) $5 \times (9 \times \frac{1}{5}) = 5 \times (\frac{1}{5} \times 9)$

(b) $5 \times (\frac{1}{5} \times 9) = (5 \times \frac{1}{5}) \times 9$

(c) $(1 \times 9) = 9$

Mr. Marston and Miss McCoy suggested that Exercise 3 as written is confusing to the students. They suggest this method of writing it:

a) $\{[5 \times (3+0)] + 0\} \times 1 = \{[5 \times 3] + 0\} \times 1$

b) $\{[5 \times 3] + 0\} \times 1 = \{5 \times 3\} \times 1$

c) $\{5 \times 3\} \times 1 = 5 \times \{3 \times 1\}$

d) $5 \times \{3 \times 1\} = \{5 \times 3\}$ or, 5×3.

* * *

Some teachers suggested that there was need for more exercises similar to those in C. Here are some you may want to use.

1. $35 \times 98\%$
2. $\frac{72}{100} \times 205$
3. $(\frac{7}{12} + \frac{5}{9}) \times \frac{5}{6}$
4. $6 \times (73\frac{2}{3})$
5. $25\frac{1}{3} \times 4\frac{2}{5}$
6. 59×0.8
7. $\frac{75}{100} \times 89$
8. 999×731
9. $972.75 \times 37 + 27.25 \times 37 + 490$
10. $(29 \times 51) + (62 \times 51) + (9 \times 51)$
11. $\frac{1}{12} \times (\frac{6}{7} + \frac{15}{9} - \frac{4}{13})$
12. $(43 \times 8.376) + (3.124 \times 43) + 37.2$

2. Simplify: $(43 \times 37) + (37 \times 20)$

   (a) $(43 \times 37) + (37 \times 20) = (37 \times 43) + (37 \times 20)$

   (b) $(37 \times 43) + (37 \times 20) = 37 \times (20 + 43)$

3. Simplify: $\{[5 \times (3 + 0)] + 0\} \times 1$

   (a) $\{[5 \times (3 + 0)] + 0\} \times 1 = [(5 \times 3) + 0] \times 1$

   (b) $[(5 \times 3) + 0] \times 1 = (5 \times 3) \times 1$

   (c) $(5 \times 3) \times 1 = 5 \times 3$

C. Write the simplest name you can for each of the following numbers. Use as little computation as possible.

1. 6×0
2. $12\frac{1}{2} \times 1$
3. $(6 + 93) + 7$
4. $799 + (1 + 58)$
5. $(9 \times 5) \times 2$
6. $\frac{4}{5} \times (5 \times 7)$
7. $2 \times (19 + \frac{1}{2})$
8. $3 \times 6\frac{2}{3}$
9. $(13 \times 7) + (13 \times 3)$
10. $(58 \times 892) + (58 \times 108)$

DIRECTED NUMBERS AND THE PRINCIPLES OF ARITHMETIC

We know how to add and multiply directed numbers. Do the principles which hold for the numbers of arithmetic also hold for directed numbers? One way to convince yourself of the answer to this question is to check the principles using directed numbers instead of the numbers of arithmetic. You know already that the principles hold for positive numbers because computations with positive numbers correspond exactly to computations with arithmetic numbers.

For example, if the statement:

$$9 \times (8 + 3) = (9 \times 8) + (9 \times 3)$$

is true, then the statement:

$$(+9) \times [+8 + (+3)] = [+9 \times (+8)] + [+9 \times (+3)]$$

is also true. There is no need to check the principles using only positive numbers. You should check them, however, using positive numbers and negative numbers together.

<u>Example</u>: Does the associative principle for multiplication apply to directed numbers?

he need not memorize these terms). Our terminology makes it possible for the student to identify each of these numbers. We think that students will accept this new type of terminology when you give them the reason for its introduction.

Similarly, we would call $12 \div 3$ <u>the quotient of 12 by 3</u>, rather than <u>the quotient of 12 and 3</u>.

But, we know that $(-1) \times (+1) = -1$. So, it should be the case that

$$[-1] + [(-1) \times (-1)] = 0.$$

Since -1 has the unique opposite +1, we must define '$(-1) \times (-1)$' to be a name for +1.

* * *

Some students have difficulty with the first sentence on page 1-47. Perhaps it should be rephrased to read:

> The associative principle for multiplication refers to the product obtained by multiplying a number by the product of a number and a number.

* * *

Notice the phrase on the next-to-last line:

the difference of 9 from 16

We introduced this terminology as a replacement for the ambiguous but commonly used phrase:

the difference between 9 and 16

Since addition and multiplication are commutative operations, phrases such as:

the sum of 4 and 7

and:

the product of 4 and 7

cause no difficulty. It doesn't matter whether the student thinks of the sum of 4 and 7 as $4 + 7$ or $7 + 4$. But since subtraction (and division, also) is not a commutative operation, it is important that the student be able to identify minuend and subtrahend (although

(continued on T. C. 47C)

Students ought to go through the checking process for each principle at least once. Use this opportunity to get in plenty of practice on adding and multiplying directed numbers if your class needs this practice. Also more maintenance work with fractions and decimals.

* * *

You may want to consider with your more able students the question of why directed numbers seem to fulfill the requirements of the principles of arithmetic. In a more rigorous treatment, the sums and products would be explained in such a way that the principles could be derived. In the present treatment, we "rigged" the interpretation of the symbols so that the students would "discover" the "correct" procedures for finding sums and products, and then verify that the principles of arithmetic held.

It is possible to use the principles of arithmetic to "guide" you in defining various symbols. For example, suppose that we want to define '$(-1) \times (-1)$', and that we have already defined '$(-1) \times (+1)$', '$(-1) \times 0$', and '$(+1) + (-1)$' in the customary fashion. We know then that

$$(-1) \times [(+1) + (-1)] = (-1) \times 0 = 0.$$

If the distributive principle is to hold for directed numbers, it should be the case that

$$(-1) \times [(+1) + (-1)] = [(-1) \times (+1)] + [(-1) \times (-1)].$$

Since the left-hand member is a name for 0, the right-hand member must also be a name for 0. That is, it should be the case that

$$[(-1) \times (+1) + [(-1) \times (-1)] = 0.$$

(Continued on T. C. 47B)

<u>Solution</u>. The associative principle refers to three numbers. Suppose the numbers are -6, +3, and -5. If the associative principle for multiplication does hold, then the following statement is true:

$[-6 \times (+3)] \times (-5) = -6 \times [+3 \times (-5)]$

We can tell whether the statement is true or not by writing simpler names for the numbers

$[-6 \times (+3)] \times (-5)$ and $-6 \times [+3 \times (-5)]$.

We can tell more readily whether two expressions are names for the same number if we can replace the expressions by simpler ones.

$$[-6 \times (+3)] \times (-5) = -18 \times (-5) = +90$$
$$-6 \times [+3 \times (-5)] = -6 \times [-15] = +90$$

The associative principle for multiplication works in this case. Try a few more cases to convince yourself.

EXERCISES

Check each of the principles illustrated on page 1-43 through 1-45 using directed numbers. For each principle check several cases.

1.11 <u>Subtracting directed numbers</u>.--Do you recall how you were taught in grade school to check subtraction problems? You were told to <u>add</u> the difference to the subtrahend. If this sum equaled the minuend, you probably subtracted correctly. See how the check is performed:

$$\begin{array}{lcrcr}
\text{minuend} & \longrightarrow & 8 & \longleftarrow & \overline{8} \\
\text{subtrahend} & \longrightarrow & -\ \underline{3} & & 3 \\
\text{difference} & \longrightarrow & 5 & & +\ 5
\end{array}$$

You learned that every subtraction problem could be checked by addition. In fact, you probably learned how to subtract numbers by thinking in terms of addition. For example, if you want to find the difference of 9 from 16:

$$16 - 9 = ?$$

Note that by agreeing to call the operation of subtraction the inverse of the operation of addition, we have completely described the operation of subtraction for directed numbers.

* * *

Again, let the student work Part A without any discussion of speedy devices. This should be a creative activity.

Some teachers expressed a need for more exercises in Part A. Here are some supplementary ones.

13. (-19) - (+8)	14. (+69) - (+91)
15. (+27) - (-6)	16. 0 - (-41)
17. (+21) - (+9)	18. (-37) - (-52)
19. (-33) - (-14)	20. 0 - (+17)
21. (-11) - (+15)	22. (-582) - (0)
23. (+21) - (-28)	24. (+771) - 0

you think:

$$9 + ? = 16$$

The fact that you can solve a subtraction problem by stating the problem as one in addition illustrates a principle called <u>the principle of inverse operations</u>. The operation of subtraction is the inverse of the operation of addition.

Now, we shall agree that the operation of subtraction <u>for directed numbers</u> is the inverse of the operation of addition for directed numbers. By thinking of subtraction in this way we can readily solve any subtraction problem.

<u>Example</u>. Find the difference of -6 from +9

<u>Solution</u>. Our problem is to solve:

$$+9 - (-6) = ?$$

Since subtraction is the inverse of addition, we can solve this problem by thinking:

$$-6 + ? = +9$$

Use your knowledge of addition to find the difference.

One way to do this is find the number which when added to -6 gives the sum 0 and then to find the number which when added to 0 gives the sum +9. The required difference is then easily obtained.

Did you find that

$$+9 - (-6) = +15 \ ?$$

EXERCISES

<u>A</u>. Subtract by using your knowledge of addition.

1. $+8 - (+3)$	2. $+8 - (-3)$
3. $-8 - (+3)$	4. $-8 - (-3)$
5. $0 - (+3)$	6. $+3 - 0$
7. $+2 - (+9)$	8. $+2 - (-9)$
9. $-2 - (+9)$	10. $-2 - (-9)$
11. $0 - (-5)$	12. $-5 - 0$

<u>B</u>. There is a quick way to solve subtraction problems. Perhaps you have discovered it by now.

Suppose we want to subtract +9 from -4:

We assume there is only one number which added to -7 gives +11; therefore, we conclude that

$$[(+11) - (-7)] = [+11 + (+7)].$$

So, our original problem may be written:

$$+11 + (+7) = ?$$

* * *

Tie in the <u>Solution</u> with the development just preceding it. You may want to repeat the development for this case. Since the development is not an easy one to reproduce, the students may ultimately agree on a rule which sounds something like "to subtract a number you add its opposite." However, avoid the expression 'change signs and add' !! [If a student volunteers an expression like this, it is not necessary that he be chastised. Just let him know that you understand what he means, but do not ask him to repeat it for the "benefit" of the rest of the class. Regard it as a private short-cut.]

We assume there is only one number which added to +8 gives 3; therefore, we conclude that

$$[+3 - (+8)] = [+3 + (-8)].$$

Hence, our original problem may be written:

$$+3 + (-8) = ?$$

II.

$$+11 - (-7) = ?$$

We know that '$[+11 - (-7)]$' names a number which must be added to -7 to obtain +11. That is:

(A) $(-7) + [+11 - (-7)] = +11.$

Since $(-7) + (+7) = 0$, then we can write:

(B) $+11 + [(-7) + (+7)] = +11.$

Applying the associative principle for addition, we have:

$$[+11 + (-7)] + (+7) = +11.$$

Then using the commutative principle for addition, we obtain:

$$[(-7) + (+11)] + (+7) = +11.$$

Again applying the associative principle for addition, we get:

(B_1) $(-7) + [(+11) + (+7)] = +11.$

Compare (B_1) with A:

(A) $(-7) + [(+11) - (-7)] = +11.$

(B_1) $(-7) + [(+11) + (+7)] = +11.$

(continued on T. C. 49D)

We assume there is only one number which added to +9 gives -4; therefore, we conclude that

$$[(-4) - (+9)] = [(-4) + (-9)].$$

Hence in any problem we may write

'[-4 + (-9)]' instead of '[-4 - (+9)]'.

* * *

If you feel it necessary to use more than one example of this proof, here are two others you may find helpful:

I.

$$+3 - (+8) = ?$$

'[+3 - (+8)]' names a number which must be added to (+8) in order to obtain +3. That is:

(A) $(+8) + (+3 - (+8)] = +3.$

Since $[(+8) + (-8)] = 0$, then we can write

(B) $+3 + [+8 + (-8)] = +3.$

Applying the associative principle for addition, we have:

$$[+3 + (+8)] + (-8) = +3.$$

Then using the commutative principle, we can write:

$$[+8 + (+3)] + (-8) = +3.$$

Again applying the associative principle, we obtain:

(B_1) $+8 + [+3 + (-8)] = +3.$

Comparing (B_1) with (A), we see

(A) $+8 + [+3 - (+8)] = 3,$

(B_1) $+8 + [+3 + (-8)] = 3.$

(continued on T. C. 49C)

According to reports from our participating teachers, there are few students who are able to follow the development as given in the text. So, work it out slowly at the board, letting the students supply the reasons for each step in the development. Perhaps this different "arrangement" of the development may be of help too.

$$-4 - (+9) = ?$$

We know [because subtraction is the inverse of addition] that '$[(-4) - (+9)]$' names a number which must be added to +9 to obtain -4. That is:

(A) $(+9) + [(-4) - (+9)] = -4.$

Since $[(+9) + (-9)] = 0$, then:

(B) $(-4) + [(+9) + (-9)] = -4.$

Applying the associative principle for addition to statement (B), we have:

$$[(-4) + (+9)] + (-9) = -4.$$

Applying the commutative principle for addition, we get:

$$[(+9) + (-4)] + (-9) = -4.$$

Again applying the associative principle for addition, we get:

(B_1) $(+9) + [(-4) + (-9)] = -4.$

Now compare (B_1) with (A):

(A) $(+9) + [(-4) - (+9)] = -4,$

(B_1) $(+9) + [(-4) + (-9)] = -4.$

(continued on T. C. 49B)

$$-4 - (+9) = ?$$

You know that $-4 - (+9)$ is the number which must be added to +9 to obtain -4. That is:

(A) $$+9 + [-4 - (+9)] = -4$$

Since $+9 + (-9) = 0$ you also know:

(B) $$-4 + [+9 + (-9)] = -4$$

Let us apply the associative and commutative principles for addition to statement (B).

$$-4 + [+9 + (-9)] = -4$$

$$[-4 + (+9)] + (-9) = -4$$

$$[+9 + (-4)] + (-9) = -4$$

(C) $$+9 + [-4 + (-9)] = -4$$

Now compare statements (A) and (C) above:

(A) $$+9 + [-4 - (+9)] = -4$$

(C) $$+9 + [-4 + (-9)] = -4$$

We assume that there is only one number which added to +9 gives -4 and so we conclude:

$$-4 - (+9) = -4 + (-9)$$

Therefore, in any problem we may write

'$-4 + (-9)$' instead of '$-4 - (+9)$'.

Now, we have a very easy method for finding certain differences.

Sample. Find the difference of +7 from +2.

Solution. The difference is written:

$$+2 - (+7)$$

and this expression stands for the same number as does the expression:

$$+2 + (-7).$$

But,

$$+2 + (-7) = -5.$$

Therefore,

$$+2 - (+7) = -5.$$

Write the simplest expression you can for each of the following using the principle illustrated above.

1. $+3 - (+9)$
2. $+12 - (+1)$
3. $-6 - (+14)$
4. $-7 - (+8)$
5. $0 - (+5)$
6. $+2 - 0$

[After students have arrived at the informal rule that "to subtract a number is the same as adding its opposite", they might be interested in the analogous situation for division--i.e., "to divide by any number (except 0) is the same as multiplying by its reciprocal".]

Here is a development for this rule:

$$3 \div 5 = ?$$

We know [because multiplication and division are inverse operations] that '$(3 \div 5)$' names a number which, when multiplied by 5, gives 3. That is:

(A) $(3 \div 5) \times 5 = 3.$

We know by the principle of reciprocals that $5 \times \frac{1}{5} = 1$

Hence we can write:

(B) $3 \times (5 \times \frac{1}{5}) = 3.$

Applying the commutative principle for multiplication, we obtain:

$$3 \times (\frac{1}{5} \times 5) = 3.$$

Then applying the associative principle for multiplication, we get:

(B_1) $(3 \times \frac{1}{5}) \times 5 = 3.$

Compare (A) with (B_1):

$$(3 \div 5) \times 5 = 3,$$
$$(3 \times \frac{1}{5}) \times 5 = 3.$$

We assume there is only one number which, when multiplied by 5, gives 3; therefore, we conclude that

$$3 \div 5 = 3 \times \frac{1}{5}.$$

C. Find a simple name for the difference -8 - (-7). Use the method shown in discussion preceding the exercises of Part B. Then use this easy way to write a simpler expression in place of each of the following expressions.

1. +3 - (-9)
2. +12 - (-1)
3. -6 - (-14)
4. -7 - (-8)
5. 0 - (-5)
6. +2 - 0

D. The ways you learned in Parts B and C of writing simple names for differences are illustrated by the following statements:

$$+7 - (-6) = +7 + (+6) = +13$$
$$+7 - (+6) = +7 + (-6) = +1$$

The name '+13' is simpler than the name '+7 - (-6)'. You get the name '+13' by recognizing that

$$+7 - (-6) = +7 + (+6)$$

and that

$$+7 + (+6) = +13.$$

Write the simplest name you can for each of the differences listed.

1. (-4) - (+8)
2. (-3) - (+6)
3. (+9) - (-2)
4. (-17) - (-8)
5. (-25) - 0
6. 0 - (-7)
7. (+7) - 0
8. 0 - (+9)
9. (-37) - (-37)
10. (37) - (+37)
11. (-16) - (-15)
12. (+2) - (+7)
13. 2 - (-2)
14. 1 - (-1)
15. 2 - (+2)
16. 1 - (+1)
17. (+7) - 2
18. (-7) - 2
19. 5 - 6
20. 7 - 21

21. $\begin{array}{r} +6 \\ -\ \ -3 \\ \hline \end{array}$ 22. $\begin{array}{r} -2 \\ -\ \ 8 \\ \hline \end{array}$ 23. $\begin{array}{r} 0 \\ -\ \ -8 \\ \hline \end{array}$ 24. $\begin{array}{r} 9 \\ -\ \ +7 \\ \hline \end{array}$

(continued on next page)

In Part E we want the student to struggle in finding continued sums. We hope that as a result of his struggling, he will be ready to accept the "grouping" shortcuts of Part F.

25. $\begin{array}{r} 12 \\ -\underline{\quad 18} \end{array}$

26. $\begin{array}{r} 4\frac{1}{2} \\ -\underline{-6\frac{1}{2}} \end{array}$

27. $\begin{array}{r} -3\frac{1}{2} \\ -\underline{-2\frac{1}{2}} \end{array}$

28. $\begin{array}{r} 0 \\ -\underline{1\frac{1}{4}} \end{array}$

29. $\begin{array}{r} 3\frac{1}{8} \\ -\underline{-2\frac{1}{4}} \end{array}$

30. $\begin{array}{r} -1.8 \\ -\underline{-2.9} \end{array}$

E. In each of the following exercises you are to simplify by proceeding from left to right. No symbols of grouping have been used. If symbols of grouping were used, Exercise 1, for example, would look like:

$$\{[(+8) - (-5)] + (-6)\} - (+12)$$

Obtain as simple a name as possible.

1. (+8) - (-5) + (-6) - (+12)
2. (+12) - (+6) - (-7) + (-8) + (-13) - (-7)
3. (-3) + (-6) - (-4) + (-7) - (-7) + (-13)
4. 3 + (-7) + 3 - (-7) - 5 + 12 + (-3)
5. (-4) + 0 + (-8) - (-3) + 11 + (-17) + 16
6. 7 - 3 + (-3) - (+8) + 17 - (-1) - (-6) + 4
7. (-10) + (-7) + (-3) - (-10) - (-7) - (-3) - (-1)
8. (-10) - (-10) + (-10) + (-3) - (-3) + 0 + (+2) - (+2)
9. (-19) + (-3) - (-4) + (-6) + 17 + (-4) - (-3) - (-19)
10. 0 - 7 + (-3) + 7 - (-8) + 9 - (-10) + (-12) + (-7)

F. In an earlier section you learned that in adding directed numbers the order in which you wrote their numerals had no effect upon the sum. For example, 7 + 3 + 5 is the same number as 3 + 5 + 7. This is not the case when you subtract directed numbers. For example,

When considering the fact that the associative principle does not apply in subtraction, you may want the students to consider this case:

$$\begin{array}{rcl} [+9 - (+6)] - (+2) & \stackrel{?}{=} & +9 - [(+6) - (+2)] \\ [9 + (-6)] + (-2) & & +9 - [(+6) + (-2)] \\ 3 + (-2) & & +9 - [+4] \\ 1 & \neq & +5 \end{array}$$

***

Students show considerable mechanical skill in exercises like those at the bottom of the page. We think they ought to try to understand that long expressions containing + and - signs can be written as expressions in which the only operation sign is ' + '. Naturally, it is not a shortcut to <u>rewrite</u> the expression in this manner. What is important is that the students understand what principles are at work when they perform the shortcut.

$$6 - 3 \neq 3 - 6$$

and

$$7 - 2 + 9 \neq 2 - 7 + 9$$

However, it is possible to rearrange the numerals in a subtraction problem if you first express the problem as one in addition. Study these illustrations:

$$6 - 3 = 6 + (-3) = (-3) + 6$$

$$8 - 3 + 7 - 2 = 8 + (-3) + 7 + (-2) = 8 + 7 + (-3) + (-2)$$

There appears to be little gained by going through the rearrangements illustrated in the examples above. However, these rearrangements are the basis for a valuable shortcut in adding and subtracting directed numbers. Any arrangement of numerals separated by addition signs and subtraction signs can be viewed as an addition. For example,

$$2 - 5 + 6 - 3 + 9 - 6 - 5$$

is the same as

$$2 + (-5) + 6 + (-3) + 9 + (-6) + (-5).$$

Then, one simply adds the positive numbers, adds the negative numbers, and adds the two sums.

<u>Sample</u>. $2 - 5 + 6 - 3 + 9 - 6 - 5$

$2 + 6 + 9 = 17$

$-5 + (-3) + (-6) + (-5) = -19$

$17 + (-19) = -2$

Simplify as illustrated above by first converting to addition.

1, $3 - 5 + 6 - 7 + 9 + 8 - 3$

2. $0 + 5 - 6 + 8 - 0 + 9 - 5$

3. $2 - 7 + 8 - 6 + 4 + 3 + 2$

4. $1 - 1 + 2 - 2 + 3 - 6 + 7$

5. $10 - 8 + 7 - 5 + 6 + 8$

6. $5 - 15 - 20 + 18 + 2 - 8$

7. $6 + 4 - 3 + 12 + 10 - 16$

8. $4\frac{1}{2} + 7 - 3\frac{1}{2} - 8\frac{1}{2} \div 9\frac{1}{2}$

(continued on next page)

9. $6\frac{1}{4} - 2\frac{1}{2} - 5\frac{1}{2} + 3\frac{1}{4} - 2\frac{1}{2}$

10. $5\frac{2}{3} - 10\frac{1}{3} + 7\frac{1}{3} - 15\frac{2}{3}$

1.12 Dividing directed numbers.--You learned to subtract directed numbers by making use of the fact that subtraction is the inverse of addition. Similarly, to divide directed numbers, you will use the fact that division is the inverse of multiplication. Recall that when you first learned to divide, for example, 6 by 3 you asked yourself:

$$? \times 3 = 6$$

Now we shall use this method for dividing directed numbers. Suppose you want to divide -20 by +5. That is, you want to simplify:

$$(-20) \div (+5).$$

You ask yourself:

$$? \times (+5) = -20$$

And from your knowledge of multiplication, the result is -4. Study the following examples.

1. $(+18) \div (+3) = ?$

   $(+18) \div (+3) = +6$ because $(+6) \times (+3) = +18$

2. $(+18) \div (-3) = ?$

   $(+18) \div (-3) = -6$ because $(-6) \times (-3) = +18$

3. $(-18) \div (+3) = ?$

   $(-18) \div (+3) = -6$ because $(-6) \times (+3) = -18$

4. $(-18) \div (-3) = ?$

   $(-18) \div (-3) = +6$ because $(+6) \times (-3) = -18$

WAYS OF NAMING A QUOTIENT

A division problem such as:

divide 8 by -2

is commonly expressed in either of two ways:

$$(+8) \div (-2) = ?$$

or:

If we admitted <u>all</u> of these statements then we would have an operation which does not give a unique result. We don't permit our operations to behave in this fashion, because if we did, we would also have to accept statements such as:

$$7 = 1, \quad 1 = 3, \quad 7 = 3, \quad \ldots$$

But suppose we admitted only one of these statements, say:

$$\frac{0}{0} = 1.$$

That is, suppose we defined '$\frac{0}{0}$' to stand for 1. Then, one immediate problem which would face us is the resolution of the following difficulty:

If $\frac{0}{0} = 1$ then, since $1 \times 8 = 8$,

$\frac{0}{0} \times 8 = 8$ also. But, $\frac{0}{0} \times 8 = \frac{0 \times 8}{0} = \frac{0}{0} = 1.$

So, $8 = 1$.

Thus, we get into trouble if we define '$\frac{0}{0}$' to stand for a number. Hence, we don't define it.

* * *

If your experience is at all typical, you will find that even after giving a very careful discussion of the problem of division by 0, students will still repeat their old errors. So, be prepared to review the discussion several times during the year. Also, you may even find some students who declare, "We agreed that we don't divide by 0, so '$\frac{0}{6}$' is an undefined mark."

As in the case of multiplication, it is not necessary for students to state a "rule of signs" for division. Proficiency in the application of a rule which they have formulated at a "sub-verbal" level is all that we should expect. You may have to supply more exercises than those found in Parts B and C in order for the students to attain this proficiency.

***

In Exercise 11 of Part B you should treat in some detail the question of why we do not divide by 0. The matter appears again in Parts H and I on pages 1-57 and 1-58.

You may start by asking,

"Does '$\frac{3}{0}$' name a number?"

Since we have agreed that division is the inverse of multiplication, we know that if the answer is 'yes', then there must be a number whose product by 0 is 3. By the principle of 0 for multiplication, we know that the product of any number by 0 is 0. Hence, '$\frac{3}{0}$' is not a numeral. So, if you want to preserve the principle of 0 for multiplication [and the explanation of what division is], you must refrain from dividing non-zero numbers by 0.

Then take these:

$$\frac{0}{0} = 7; \ \frac{0}{0} = 1; \ \frac{0}{0} = 3; \ \ldots$$

All of these "check" by multiplication.

(continued on T. C. 54B)

In Parts D and E we introduce without explanation a notation which is quite useful. You should supply as much explanation as necessary. We can write a name for the opposite of a given number simply by writing a minus sign at the left of the name of the given number. We frequently enclose the given name in parentheses before we write the minus sign. Consider Exercise 2 of Part D:

> The number -(5 + 2) is the opposite of 5 + 2. But 5 + 2 is 7. So, -(5 + 2) is -7. In (b), we know that -1 × (5 + 2) is -1 × 7 or -7.
>
> We also know that
>
> $$\begin{aligned} -1 \times (5 + 2) &= (-1 \times 5) + (-1 \times 2) \\ &= -5 + (-2) \\ &= -5 - 2. \end{aligned}$$
>
> Therefore,
>
> $$-(5 + 2) = -5 - 2.$$

Thus, in Exercises 1, 2 and 3 we have a basis for the process of "removing parentheses preceded by a minus sign". The only thing we want to stress <u>now</u> is that

> -(5 + 2) is equal to -5 - 2 because
> -(5 + 2) and -1 × (5 + 2) are equal.

1. $(+12) \div (+3)$
2. $(-17) \div (+1)$
3. $(-6) \div (-2)$
4. $(+8) \div (-2)$
5. $(+10) \div (-1)$
6. $(-7) \div (-1)$
7. $0 \div (-3)$
8. $(+9) \div (+3)$
9. $(+16) \div (-4)$
10. $17 \div (-1)$
11. $\frac{-16}{8}$
12. $\frac{-21}{-7}$
13. $\frac{-33}{3}$
14. $\frac{0}{-17}$
15. $\frac{+34}{-17}$
16. $(-18) \div (-6)$
17. $18 \div (-3)$
18. $18 \div 3$
19. $(+18) \div (+3)$
20. $(+4) \div (+1)$
21. $(-4) \div (-1)$
22. $\frac{-4}{+1}$
23. $\frac{+4}{-1}$
24. $\frac{4}{-1}$
25. $(+4) \div 1$
26. $4 \div 1$
27. $6 \div (-3)$
28. $(-7) \div 1$
29. $0 \div (-8)$
30. $(-15) \div 3$

D. Simplify the expression in each of the following exercises.

1. (a) $-(-5)$
   (b) $-1 \times (-5)$
2. (a) $-(5 + 2)$
   (b) $-1 \times (5 + 2)$
3. (a) $-(5 - 7 + 1)$
   (b) $-1 \times (5 - 7 + 1)$
4. (a) $-(-3 \div 6)$
   (b) $-1 \times (-3 \div 6)$
5. (a) $-\frac{-8}{4}$
   (b) $-1 \times \frac{-8}{4}$

Compare the opposite of a number with the result of multiplying it by -1.

E. Simplify the expressions in each of the following exercises.

(continued on next page)

The phrase 'invert and multiply' cannot be taken literally, since one can no more invert a number than he can pat it on the back. It is in the same category as 'change the sign and add'. Both of these are excellent as "private" shortcuts, where the individual who has <u>discovered</u> the shortcut for himself knows exactly what its limitations are. To <u>give</u> these shortcuts to a class of students when all the students do not fully understand the limitations is to invite mistakes such as these:

$$(+7) - (-3)(+4) = (+7) + (+3)(-4)$$

$$\frac{7}{\frac{2}{3} + \frac{3}{5}} = 7 \times \left(\frac{3}{2} + \frac{5}{3}\right)$$

For example:

$$3 \div \frac{3}{2} \left[= \frac{3}{\frac{3}{2}} \right] = \frac{3 \times \frac{2}{3}}{\frac{3}{2} \times \frac{2}{3}} = \frac{3 \times \frac{2}{3}}{1} = 3 \times \frac{2}{3}$$

A parallel development for subtraction depends upon the notion that if you add the same number to each of two numbers, the difference of the second number from the first number is the same as the difference of the second sum from the first sum. For example:

$$\begin{aligned} (+4) - (+3) &= [(+4) + (-3)] - [(+3) + (-3)] \\ &= [(+4) + (-3)] - [0] \\ &= (+4) + (-3) \end{aligned}$$

The last step in the first example:

$$\frac{3 \times \frac{2}{3}}{1} = 3 \times \frac{2}{3},$$

is justified by the explanation of division as the inverse of multiplication, and the principle of 1 for multiplication. $\left[(3 \times \frac{2}{3}) \div 1 \right]$ is the number whose product by 1 is $3 \times \frac{2}{3}$. But, this number is $3 \times \frac{2}{3}$. [Similarly, in the second example, $\{[(+4) + (-3)] - [0]\}$ is the number whose sum with 0 is $[(+4) + (-3)]$. But this number is $[(+4) + (-3)]$.]

***

Note in Part F the parallelism between pairs of opposites and pairs of reciprocals. The sum of a pair of opposites is 0; the product of a pair of reciprocals is 1. This parallelism is very useful in pointing out how the notion of inverse operations comes into play in subtracting and dividing. In Part G [page 1-57] the student reviews the fact that "dividing by a number gives the same result as multiplying by the reciprocal of the number".

We derive an instance of the rule:

(A) $(3 \div \frac{3}{2}) \times \frac{3}{2} = 3$ [division is the inverse of multiplication]

$\frac{2}{3} \times \frac{3}{2} = 1$ [principle of reciprocals]

$3 \times \left(\frac{2}{3} \times \frac{3}{2}\right) = 3 \times 1 = 3$ [principle of 1 for multiplication]

(B) $\left(3 \times \frac{2}{3}\right) \times \frac{3}{2} = 3$ [associative principle for multiplication]

Comparing (A) and (B) you see:

$3 \div \frac{3}{2} = 3 \times \frac{2}{3}$ [the result of division is unique]

The rule "to divide by a number, you multiply by its reciprocal" is analogous to the rule "to subtract a number, you add its opposite".

***

Sometimes the following development "makes sensible" the business of "multiplying by the reciprocal". It depends upon the notion that the quotient of two numbers is the same as the quotient of their products by any number other than 0 ["cancellation law"].

1. (a) $-(4 - 1)$
   (b) $(4 - 1) \div (-1)$

2. (a) $-(3 \div 2)$
   (b) $(3 \div 2) \div (-1)$

3. (a) $-[5 \times (-2\frac{1}{5})]$
   (b) $[5 \times (-2\frac{1}{5})] \div (-1)$

4. (a) $-[-18 \div (-6)]$
   (b) $[-18 \div (-6)] \div (-1)$

5. (a) $-[42 \div (-\frac{5}{3})]$
   (b) $[42 \div (-\frac{5}{3})] \div (-1)$

Compare the opposite of a number with the result of dividing the number by -1.

F. Adding or subtracting 0 from a number leaves the number unchanged. You are familiar with the fact that for every directed number there is another directed number which added to it gives 0. Thus, $(-3) + (+3) = 0$, $+4.567 + (-4.567) = 0$, etc. The number -3 is called the <u>opposite of the number +3</u>, and the number -4.567 is called the opposite of the number +4.567, etc.

Multiplying or dividing a number by 1 leaves the number unchanged also. For every directed number except 0 there is another directed number which when multiplied by the first number gives 1. Each of these numbers is called the <u>reciprocal of the other</u>. For example,

since $3 \times \frac{1}{3} = 1$, $\frac{1}{3}$ is called the <u>reciprocal of 3</u> and 3 is called the <u>reciprocal of $\frac{1}{3}$</u>;

since $-\frac{5}{3} \times (-\frac{3}{5}) = 1$, $-\frac{5}{3}$ is called the <u>reciprocal of $-\frac{3}{5}$</u> and $-\frac{3}{5}$ is called the <u>reciprocal of $-\frac{5}{3}$</u>.

In each of the following exercises, find the reciprocal of the number listed.

1. 3
2. $\frac{1}{4}$
3. 5
4. $\frac{2}{3}$
5. -4
6. $-\frac{1}{3}$

(continued on next page)

Check the following statements and tell why they are right or why they are wrong.

1) '0' divided by any number is 0.

2) Any positive number divided by 1 is itself.

3) A number multiplied by itself is twice that number.

4) The sum of two numbers is either greater than or smaller than their product. [Wrong: $\frac{7}{6}$ and 7, $\frac{9}{8}$ and 9, . . .]

5) A number divided by itself is 1. Therefore, 0 divided by 0 is 1.

6) 0 multiplied by any number is 0 . Therefore, 0 divided by any number is 0. I can prove this by the principle that division is the inverse of multiplication.

7) 0 divided by 3 is 0; 0 divided by 5 is 0. Therefore, 3 = 5.

So we can write:

(A) $\left[\frac{3}{5} \div \frac{7}{8}\right] \times \frac{7}{8} = \frac{3}{5}$.

(B) $\frac{3}{5} \times \left[\frac{7}{8} \times \frac{8}{7}\right] = \frac{3}{5}$.

Since multiplication is commutative, we can write:

$$\frac{3}{5} \times \left[\frac{8}{7} \times \frac{7}{8}\right] = \frac{3}{5}.$$

Applying the associative principle for multiplication, we get:

(B_1) $\left[\frac{3}{5} \times \frac{8}{7}\right] \times \frac{7}{8} = \frac{3}{5}$.

Compare (A) with (B_1):

(A) $\left[\frac{3}{5} \div \frac{7}{8}\right] \times \frac{7}{8} = \frac{3}{5}$,

(B_1) $\left[\frac{3}{5} \times \frac{8}{7}\right] \times \frac{7}{8} = \frac{3}{5}$.

Since there is only one number which when multiplied by $\frac{7}{8}$, gives $\frac{3}{5}$, we conclude that

$$\frac{3}{5} \div \frac{7}{8} = \frac{3}{5} \times \frac{8}{7}.$$

In connection with the exercises in Part I, here are some supplementary ones that may help get across correct ideas in regard to 0 and 1.

(continued on T. C. 57C)

***

If you discussed division by 0 in connection with the work on page 1-54 [see T. C. 54A *f*], then Exercises 6 through 10 will serve as a review. Be sure that the student does not come away with the feeling that '0 ÷ 0' is not defined because it is "too hard to do". This sometimes happens when students are told, "You can't divide by 0." The student should understand that we choose not to define '0 ÷ 0' to stand for a number because such a definition would contradict other definitions in our arithmetic.

***

If you need another illustration for the rule "to divide by a number, you multiply by its reciprocal", you might use this:

$$\frac{3}{5} \div \frac{7}{8} = ?$$

a) We know that '$\left[\frac{3}{5} \div \frac{7}{8}\right]$' names only one number which, when multiplied by $\frac{7}{8}$, gives $\frac{3}{5}$. (Division is the inverse of multiplication.)

b) Also, we know that $\frac{7}{8} \times \frac{8}{7} = 1$, by the principle of reciprocals.

(continued on next page)

7. $\frac{4}{7}$ 8. $\frac{-3}{8}$ 9. $\frac{28}{27}$

10. $\frac{-432}{271}$ 11. $\frac{-214}{6}$ 12. $\frac{5001}{3001}$

G. Simplify the expression in each of the following exercises.

1. (a) $3 \div \frac{3}{2}$

   (b) $3 \times \frac{2}{3}$

2. (a) $-4 \div \frac{8}{4}$

   (b) $-4 \times \frac{4}{8}$

3. (a) $\frac{4}{3} \div \frac{6}{12}$

   (b) $\frac{4}{3} \times \frac{12}{6}$

4. (a) $\frac{1}{-8} \div \frac{2}{9}$

   (b) $\frac{1}{-8} \times \frac{9}{2}$

Compare the result of multiplying a number by a second number with the result of dividing the number by the reciprocal of the second number. Explain the rule, ''invert and multiply'' for dividing by a fractional number.

H. Rewrite as statements concerning division each of the following statements concerning multiplication.

Sample. $6 \times (-3) = -18$

Solution. $-18 \div (-3) = 6$

From your attempted ''answers'' to exercises 6 through 10, explain why division by 0 is impossible.

1. $4 \times 2 = 8$
2. $81 \times 9 = 729$
3. $5 \times (-4) = -20$
4. $-2.1 \times 10 = -21$
5. $-\frac{3}{2} \times (-\frac{6}{3}) = 3$
6. $5 \times 0 = 0$
7. $2 \times 0 = 0$
8. $-10 \times 0 = 0$
9. $\frac{3}{5} \times 0 = 0$
10. $-\frac{8}{11} \times 0 = 0$

I. Simplify each of the following expressions to review your knowledge of 0 and 1.

1. $6 + 0$
2. $-7 + 0$
3. $0 - 5$
4. 0×0
5. 0×2
6. 3×0

(continued on next page)

This section on comparing numbers is designed to prepare students for the number line concept in the section which follows. We are trying to develop here the idea that you can tell the larger of two numbers by finding their difference. If their difference is a positive number, then the minuend is greater than the subtrahend. If their difference is a negative number, than the minuend is less than the subtrahend.

In order to develop this principle for the student, we first show them that it is in agreement with their notion of comparison gained in arithmetic. Is $\frac{5}{8}$ greater than $\frac{9}{15}$? It is, if you can subtract $\frac{9}{15}$ from $\frac{5}{8}$ and obtain a difference which is not 0. It is not, if you can't subtract $\frac{9}{15}$ from $\frac{5}{8}$.

7. $\frac{3}{2} \times 1$

8. $\frac{4}{-3} \times 1$

9. $0 \div 5$

10. $5 \div 0$

11. $0 \div \frac{14}{3}$

12. $\frac{14}{3} \div 0$

1.13 Comparing numbers.--In arithmetic if you were given two different numbers, you could compare them and tell which of them was the larger. If the numbers were whole numbers, say 12 and 38, you knew that 38 was larger than 12. You may have decided this by thinking quickly of, say 38 books and 12 books or you may simply have remembered that in counting, 1, 2, 3, etc., 38 comes after 12.

There is another way to compare two numbers. Suppose someone tells you that he is thinking of two numbers of arithmetic and that if he subtracts the first number from the second number, he gets 5. Can you tell which of his two numbers is the larger? You can use this idea to decide which of two numbers, say, $\frac{9}{15}$ and $\frac{5}{8}$, is the larger. Try to subtract $\frac{5}{8}$ from $\frac{9}{15}$:

$$\frac{9}{15} - \frac{5}{8} = \frac{72}{120} - \frac{75}{120} = \frac{72 - 75}{120}$$

Note that this difference is not a number of arithmetic, and that, therefore, $\frac{9}{15}$ is not larger than $\frac{5}{8}$. In fact, you learn from this that $\frac{5}{8}$ is larger than $\frac{9}{15}$.

Numbers are so frequently compared in mathematics that we need abbreviations for 'is greater than' (or 'is larger than') and for 'is less than' (or 'is smaller than'). We shall write '>' in place of 'is greater than' and '<' in place of 'is less than'. Thus, '5 > 2' is a true statement. However, '3 < 2' is a false statement and '2 < 2' is a false statement.

Of course, if the difference is 0 then you weren't comparing <u>two</u> numbers.

***

For the most part, conventional courses in ninth grade mathematics, do not include work with the symbols of inequality. We think this is an error of omission. For by acquainting students with these symbols, we give ourselves a great deal of freedom later in the course in constructing graphing exercises, and in developing the ideas underlying the process of solving equations. Moreover, when students become acquainted with these new symbols, it becomes easier to communicate certain ideas.

In these exercises the student is expected to support his claim by using the subraction criterion. For example, consider Exercise 2 of Part B. The student should write 'False'. When asked "Why ?" he should respond by saying:

> Because in grade-school arithmetic, I can't subtract 6 from 5.

***

Note that the student ought to be able to recognize that '6 < 5' and '5 > 6' mean the same thing.

***

The student understands that when he restricts himself to grade-school arithmetic, he has a very neat criterion for comparing two numbers. If he can subtract the first number from the second number, then the second number is greater than the first number. If he can't subtract the first number from the second number, then he knows that the first number is greater than the second number. His decision hinges on whether he can subtract or not. Now, in comparing directed numbers, he can no longer use this neat criterion because <u>subtraction</u> <u>is</u> <u>always</u> <u>possible</u>. So, we state a new criterion:

If the difference of the first number from the second number is a positive number, then the second number is greater than the first number.

If the difference of the first number from the second number is a negative number, then the second number is less than the first number.

(continued on T. C. 59C)

The exercises in Part A are intended to give students practice in the use of the new symbols '<' and '>'. They may use whatever intuitive ideas they have about comparison in order to carry out these exercises. In comparing fractional numbers, it is customary to write fraction names having the same denominator. An interesting variation is to compare by writing fraction names having the same numerator.

Miss Wandke said that Exercise 8 in Part A caused some difficulty. You may want to anticipate this by having the class consider a name other than '$\frac{2}{.3}$' for the number $\frac{2}{.3}$. Someone may suggest: $\frac{2}{\frac{3}{10}}$.

Then the students may get a "simpler" name by thinking of the "multiplying by reciprocal" rule for division. Thus:

$$\begin{aligned} 2 \div \frac{3}{10} &= 2 \times \frac{10}{3} \\ &= \frac{20}{3}. \end{aligned}$$

***

The exercises in Part B are the first of a series of true-false exercises in this unit. We are trying to prepare students for Unit 2 by developing the notion that a written statement can be either true or false. Therefore, students should <u>not</u> attempt to alter the false statements so that they become true statements.

(continued on T. C. 59B)

A. In each of the following exercises two numerals are written. Copy the exercise and insert '<', '>' or '=' so that the result is a true statement.

Sample. 6 1

Solution. 6 > 1

1. 5 2 2. 3 3 3. 16 143

4. $5\frac{9}{10}$ $6\frac{1}{10}$ 5. $4\frac{3}{12}$ $4\frac{1}{4}$ 6. $\frac{1}{8}$ $\frac{1}{7}$

7. $\frac{1}{432}$ $\frac{1}{441}$ 8. $\frac{1}{0.1}$ $\frac{1}{0.2}$ 9. $\frac{32}{65}$ $\frac{33}{64}$

B. Tell whether each of the following statements is true or false.

1. $5 > 2$ 2. $6 < 5$ 3. $3.16 > 3.16$

4. $1,407 < 1,470$ 5. $\frac{1}{3} > \frac{1}{9}$ 6. $\frac{1}{6} < \frac{1}{5}$

COMPARING DIRECTED NUMBERS

When you want to compare two numbers of arithmetic, you have a very simple device. You try to subtract one of the numbers from the other. You can tell which of the numbers is larger depending upon whether you can subtract or not. Now, in comparing directed numbers you cannot use exactly this technique because it is always possible to subtract with directed numbers. We can still use subtraction but the decision as to which number is larger depends upon whether the difference is a positive number or not. Study the following examples:

$+3 > -3$ because $+3 - (-3) = +6$ and $+6$ is positive

$+5 > -7$ because $+5 - (-7) = +12$ and $+12$ is positive

$-6 < -3$ because $-6 - (-3) = -3$ and -3 is negative

You may meet some of the standard resistence to accepting the truth of a statement such as:

$$-1000 < -2.$$

If a student objects that this "just doesn't seem right", it is because he is comparing it with '1000 > 2', and anticipating the work on absolute value which starts on page 1-64.

He should realize that the idea of "greater than" for negative numbers is a new one, and that we must construct a criterion and then agree to live with it.

Also, some students may give a nonsensical objection to the truth of, say:

$$-5 < 0,$$

arguing, "How can something be less than nothing?". Their difficulty is in calling 0 <u>nothing</u>. 0 is a perfectly good number and it is not nothing. [To say that there are 0 things in a basket is another matter; then you are saying there is nothing in the basket. But this is talking about things rather than numbers.]

In all of the exercises on pages 1-60 and 1-61, it seems to be the case that the student must use the new criterion to make decisions. We believe, however, that the students will form "a number line picture" in his head without prompting from the teacher in order to avoid applying the criterion each time. This is another example of a creative activity which students enjoy.

One of Miss McCoy's students suggested that given two negative numbers, the one which is "closer" to 0 is the larger. Naturally, the student had a type of number line image in mind which helped him make this generalization. Most students create such an image for themselves. Perhaps you can get all of your students to construct such an image by asking them how they can determine quickly the larger of two given negative numbers. Of course, don't raise this question until they have worked several exercises.

* * *

In Part C you may have to instruct your students in the use of '$\neq$' in case they have not seen this symbol in earlier grades. Notice the important difference in the way '$\ngtr$' and '$<$' are used. A student should not think that these symbols can be used interchangeably. You will find that students need to spend considerable time in thinking before they make decisions in these exercises.

* * *

(continued on T. C. 60B)

EXERCISES

A. In each of the following exercises two numerals are written. Copy the exercise and insert '$<$', '$>$', or '$=$' so that the result is a true statement.

1. $+5 \quad +3$
2. $+5 \quad +19$
3. $+\frac{5}{3} \quad 0$
4. $+4 \quad -2$
5. $-7 \quad +2$
6. $-6 \quad -\frac{18}{3}$
7. $-\frac{12}{10} \quad -\frac{7}{5}$
8. $+\frac{141}{376} \quad -\frac{48}{121}$
9. $-.0013 \quad -.024$
10. $-122,426 \quad -122,425$

B. Tell whether each of the following statements is true or false.

1. $-1000 > -2$
2. $-10 > +8$
3. $-\frac{21}{71} > -\frac{20}{71}$
4. $-\frac{14}{3} < -5$
5. $+0.016 > +0.0016$
6. $-0.016 > -0.0016$
7. $+1\frac{1}{2} < +\frac{5}{4}$
8. $\frac{-6}{-3} = +2$
9. $-\frac{16}{3} > 1$
10. $-1,425 < 0$

C. You are familiar with using '$\neq$' for 'is not equal to'. We shall also use '$\nless$' for 'is not less than' and '$\ngtr$' for 'is not greater than'. Notice that '$\ngtr$' does not have the same meaning as '$<$' because, for example, '$5 \ngtr 5$' is true and '$5 < 5$' is false. Tell whether each of the following statements is true or false.

1. $+5 \ngtr +10$
2. $+4 \nless 0$
3. $-10 \neq -20$
4. $-10 \neq -10$
5. $+\frac{1}{2} \ngtr +\frac{3}{4}$
6. $+2 \ngtr +2$
7. $+5 \ngtr +\frac{10}{2}$
8. $-3 \ngtr 0$
9. $-13 \nless +\frac{5}{3}$
10. $-3 \nless -3$

By now, some students are reading '$\leq$' as:

lessthanerequalto,

and are beginning to forget that the connective between 'less than' and 'equal to' is 'or', not 'and'. This can get them into lots of difficulty because there are no numbers such that the first is equal to the second <u>and</u> also less than the second.

The student should realize that for the statement:

It is raining or Bill has brown hair,

to be true, at least one of the statements:

(1) It is raining,

and: (2) Bill has brown hair,

needs to be true. Similarly, for the statement:

$$3 \leq 5,$$

to be true, at least one of the statements:

(1) $3 = 5$

and: (2) $3 < 5$,

needs to be true. In the example about Bill's hair and the weather, both statements could be true; in the example about 3 and 5, at most one of the statements can be true.

By now, some students are reading '$\leq$' as:

lessthanerequalto,

and are beginning to forget that the connective between 'less than' and 'equal to' is 'or', not 'and'. This can get them into lots of difficulty because there are no numbers such that the first is equal to the second <u>and</u> also less than the second.

The student should realize that for the statement:

It is raining or Bill has brown hair,

to be true, at least one of the statements:

(1) It is raining,

and: (2) Bill has brown hair,

needs to be true. Similarly, for the statement:

$$3 \leq 5,$$

to be true, at least one of the statements:

(1) $3 = 5$

and: (2) $3 < 5$,

needs to be true. In the example about Bill's hair and the weather, both statements could be true; in the example about 3 and 5, at most one of the statements can be true.

Part C on page 60 should provide good background for the work in Part D on page 61. The student who understands that '$\not>$' and '$<$' are not interchangeable should have no trouble understanding that '$\not>$' and '$\leq$' are interchangeable. Some students might object to a statement such as:

$$3 \leq 3,$$

because they feel that the statement does not tell as much as it could. On the other hand, students do accept readily the statement:

$$3 \not> 3.$$

They should understand that the statements '$3 \leq 3$' and '$3 \not> 3$' are equivalent.

(continued on T. C. 61B)

D. It may be easier for you to think, "is less than or is equal to", instead of "is not greater than". Explain why these two phrases have the same meaning. We shall abbreviate 'is less than or is equal to' by '$\leq$' or '$\leqq$'. Thus, '$5 \leq 6$' is true because 5 is less than 6. Also, '$5 \leq 5$' is true because 5 is equal to 5.

Similarly, we shall abbreviate 'is greater than or is equal to' by '$\geq$' or '$\geqq$'. Tell whether each of the following statements is true or false.

1. $+5 \geqq +6$
2. $10 \leqq 26$
3. $+4 \leq +4$
4. $+\frac{1}{2} \geq +\frac{1}{2}$
5. $+3 \geqq -2$
6. $-3 \leq -\frac{21}{2}$
7. $+1.53 \leq 1.053$
8. $-1.542 \leq +.001$
9. $-\frac{11}{7} \leq -\frac{21}{14}$
10. $-\frac{14}{33} \leqq +\frac{29}{66}$

E. In each of the following exercises two numerals are written. Copy the exercise and insert one of the symbols '$=$', '$\neq$', '$<$', '$>$', '$\leq$', '$\geq$', '$\nless$', and '$\ngtr$' so that the result is a true statement. Use other of the symbols to form as many other true statements as possible.

<u>Sample.</u> $+6 \quad +4$

<u>Solution.</u>

$+6 > +4$

$+6 \neq +4$

$+6 \geq +4$

$+6 \nless +4$

1. $+5 \quad +3$
2. $-4 \quad -4$
3. $+6 \quad -3$
4. $-10 \quad -9$
5. $-\frac{10}{3} \quad +2$
6. $+0.0102 \quad +0.0112$
7. $-\frac{1}{782} \quad 0$
8. $0 \quad +\frac{431}{271}$
9. $0 \quad 0$
10. $0 \quad \frac{-3}{-7}$

thing. No harm is done, however, if the student does identify the point with the number, provided that he doesn't think that the chalk dot or pencil dot is the number. In SECOND COURSE, students learn that when the word 'point' occurs as an undefined term in a postulational system, one of the most useful interpretations of 'point' is that it means an ordered pair of numbers.

Part F seeks to do more than be humorous. Students ought to see immediately that by inserting this symbol you get a true statement in every exercise. But this fact is important because it is one of the justifications for being able to think of numbers as associated with points on a line. Also, at your discretion, you may want to point out to the student that a statement such as:

+6 is less than _or_ is equal to _or_ is greater than -3,
is true if any _one_ of the statements:

+6 is less than -3,
+6 is equal to -3,
and: +6 is greater than -3,

is true, and that in general, any statement which is compounded of substatements connected by 'or' is true if at least one of the substatements is true. The statement is true even when some of the substatements are false. For example,

'Elephants fly _or_ ice is hot _or_ 1 + 1 = 2'

is a true statement. [For further examples of the use of the connective 'or' including the dual use of this word, see Tarski, _Introduction to Logic_, page 21.]

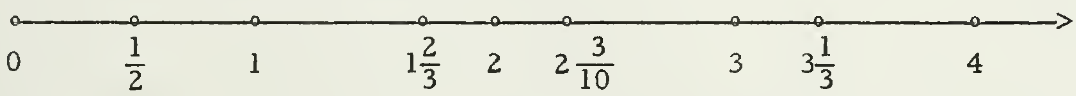
* * *

Since the student now has an order for the directed numbers, he is quite willing to accept the idea of a number line or number scale. He has probably seized upon this idea by himself. We do not think that the student will conceive of a point and a number as one and the same

(continued on T. C. 62B)

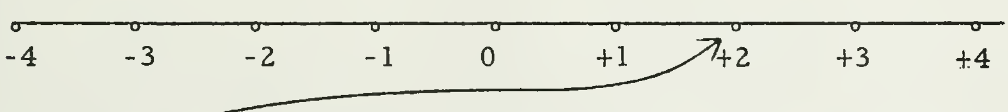
F. Suppose we abbreviated 'is less than or is equal to or is greater than' by '$\lesseqgtr$'. In what exercises in Part E would the use of '$\lesseqgtr$' lead to a true statement?

1.14 <u>The number line</u>.--You may be accustomed to thinking of the numbers of arithmetic as arranged along a line <u>in order</u>:



Of course, the numbers themselves are not points on the line but we think of the numbers as corresponding to points on the line. When the numbers of arithmetic are thought of as corresponding to the points on a line, there is an easy way to tell what is meant by, for example '2 < 3'. The expression '2 < 3' tells you that the point corresponding to 2 is on the left of the point corresponding to 3.

Since we now know how to use ' < ' and ' > ' with directed numbers, we can also think of directed numbers as arranged along a line with, for example ' < ' meaning the same as 'on the left of':



The point which corresponds to +2 is called the <u>graph</u> of the number +2 and the number +2 is called the <u>coordinate</u> of that point.

> A line consisting of points which correspond to the directed numbers is called a <u>number line</u>.
>
> Each point is the <u>graph</u> of the corresponding directed number.
>
> Each directed number is the <u>coordinate</u> of the corresponding point.

EXERCISES

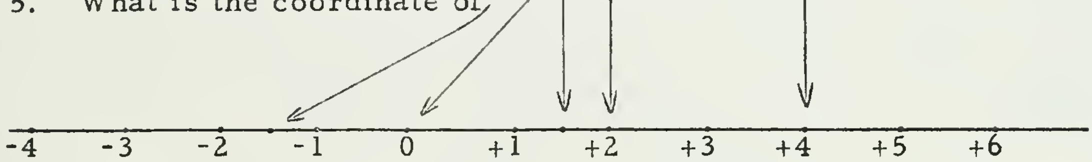
A. 1. What is the coordinate of

2. What is the coordinate of

3. What is the coordinate of

4. What is the coordinate of

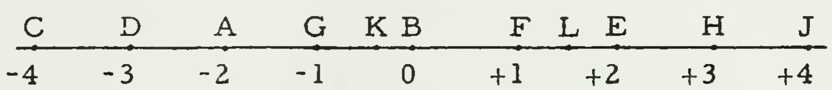
5. What is the coordinate of



B. Copy the number line and numerals near it from Part A and answer the following questions by making dots with your pencil.

1. Where is the graph of +3?
2. Where is the graph of +0.5?
3. Where is the graph of $+2\frac{1}{3}$?
4. Where is the graph of -2.5?
5. Where is the graph of -3.2?

C. It is usually inconvenient to indicate points by drawing arrows as in Part A. We will usually name points with letters:



Thus, 'A' is the name of the point with coordinate -2, and we say, that the point A has coordinate -2.

Using letters as names for points, answer the following questions.

1. What is the graph of -1?
2. What is the graph of 0?
3. What is the graph of $-\frac{1}{3}$
4. What is the graph of +3?
5. What is the graph of $1\frac{1}{2}$?

(continued on next page)

If you observe carefully our use of language, you will note that we avoid the indiscriminate use of such words as 'value' and 'quantity'. We hope that you have avoided them, also. The word 'value' is introduced in Unit 4. If you need to use 'value' before then, use it in this way:

> The value of the symbol '7' is 7.
> The value of 'IV' is 4.

When variables (pronumerals) are introduced in Unit 2, you can speak of a value as a number, one of whose names can replace a variable in an expression. For example,

> If I select the value 7 and write a numeral for it in place of 'x' in the expression:
>
> $$x + 3,$$
>
> the resulting expression has the value 10.

Most of the time, you can avoid 'value' by using 'number' instead.

The only time we use 'quantity' is when we are using in a vague way the language of the everyday physical world. 'Quantity' can be precisely defined, but we shall not do so. [Of course, we do say 'the quantity' when we read aloud:

$$7 + 3(8 + 4)$$

as: seven plus three times the quantity eight plus four.

In this case the words 'the quantity' should be considered as a signal to the listener that a parenthetical expression is involved.]

Note that the absolute value of a directed number is not a number of arithmetic. It is non-negative directed number. This point is not a settled matter among mathematicians who regard the non-negative directed numbers as different from the numbers of arithmetic. Some mathematicians regard '$|-6|$' and '$|+6|$' as names for 6, a number of arithmetic. This may be a more "natural" way; in any event it is probably the way students are inclined to regard '$|-6|$' and '$|+6|$'. It is possible to maintain this point of view and be perfectly consistent. In fact, we ourselves have not reached a final decision concerning this matter; in a future revision, we may use the other definition.

However, in the present materials, we assert that the absolute value of a directed number is not a number of arithmetic, and we maintain this position throughout our courses.

* * *

Guard against the semantic confusion inherent in the old refrain "the absolute value of a directed number is the number without regard to its sign". But even this is not as bad as "the absolute value of a directed number is its numerical value". Of course, one is free to define 'numerical value' as he pleases. But, from a pedagogical point of view, this use of 'value' is bad. Even the use of 'value' in 'absolute value' is not good, but we are stuck with a widely-used phrase.

(continued on T. C. 64B)

6. What is the coordinate of A?
7. What is the coordinate of J?
8. What is the corrdinate of C?
9. What is the coordinate of E?
10. What is the coordinate of D?

D. Your mental picture of the number line will probably make it easier for you to compare directed numbers. Thinking of their graphs complete the following exercises with '>' or '<' so that the resulting statement is true.

1. -3 +17	2. $+4\frac{1}{2}$ +3
3. -6 -5	4. -152 -2,176
5. +.0012 -.0138	6. $\frac{11}{3}$ $\frac{21}{6}$
7. -6.382 $+\frac{1}{2}$	8. +.001 +.0001
9. -1,428 +.0052	10. -.00016 -43,213

ABSOLUTE VALUE

It may have seemed strange to you when you first read, for example, that -100 was smaller than +5. Certainly if we think of these numbers as trips measured in miles, a trip corresponding to -100 takes more gasoline than a trip corresponding to +5. In order to be able to talk about such things as the amount of gasoline used, we need to learn about the absolute value of directed numbers. Study the following examples:

The absolute value of +6 is +6.
The absolute value of -6 is +6.
The absolute value of +1.52 is +1.52.
The absolute value of $-3\frac{1}{2}$ is $+3\frac{1}{2}$.
The absolute value of 0 is 0.

Usually, instead of saying, "the absolute value of ...", we enclose the name of the number we are talking about between vertical lines. Thus, the five statements above are abbreviated as:

$$|+6| = +6$$

$$|-6| = +6$$

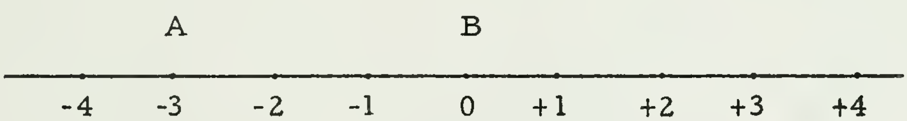
$$|+1.52| = +1.52$$

$$\left|-3\frac{1}{2}\right| = +3\frac{1}{2}$$

$$|0| = 0$$

If we are discussing the amounts of gasoline used on trips, a trip of +6 is the "same" as a trip of -6, and you have learned $|+6| = |-6|$. Do you see how the idea of absolute value is useful whenever you want to consider all trips as if they had been made in the same direction? If you think of +6 as corresponding to a trip of 6 units in the positive direction then $|+6|$ corresponds to the same trip. If -6 corresponds to a trip of 6 units in the negative direction then $|-6|$ corresponds to a trip of 6 units in the <u>positive</u> direction.

There is another way for you to think of absolute value. On the number line the absolute value of a number tells you the distance between the graph of that number and the graph of 0.



In the figure above, the distance (positive distance) between B and A is $|-3|$. The graph of 0 is usually given a special name:

> On a number line the graph of 0 is called the <u>origin</u>.
>
> The absolute value of a number tells you the distance between the graph of that number and the origin.

In Exercises 7 and 8, only one number is listed, although two numerals for that number are given.

* * *

Ask students to insert a comparison symbol in the exercises of Part C so that some of the resulting statements are false. Make a game out of this by having one student read a completed statement and having another student declare 'true' or 'false'.

* * *

Note that in Part D there is another way of looking at the statement about erasing the sign to see that it is wrong. If a student thinks incorrectly that the absolute value of a directed number is a number of arithmetic, he should be asked if he thinks that the following is true:

> "See Patricia over there near the wall. She is too tall. To make her shorter all you have to do is replace her by her baby sister."

EXERCISES

A. Tell the absolute value of each of the listed numbers.

1. $+5$ 2. -3 3. $-\frac{3}{2}$

4. $+4.51$ 5. $+\frac{27}{8}$ 6. $-4\frac{1}{3}$

7. 0 8. $+6 - (+6)$ 9. $4 - 8$

10. $3 + 7$ 11. $3 - 10$ 12. $(-3) - (-7)$

B. Answer the following questions.

1. What two numbers each have absolute value +7?
2. What two numbers each have absolute value +2?
3. What two numbers each have absolute value +527?
4. Does any number have absolute value 0?
5. Does any number have absolute value -3?

C. In each of the following exercises two numerals are given. Copy the exercise and insert '<', '>', or '=' so that the resulting statement will be true.

Sample. $|-6| \quad +4$

Solution. $|-6| > +4$

1. $|+5| \quad |+3|$ 2. $|-5| \quad |+3|$

3. $|+7| \quad |+2|$ 4. $|-7| \quad |-2|$

5. $|0| \quad |-7|$ 6. $|0| \quad +7$

7. $\left|-\frac{4}{10}\right| \quad |+0.4|$ 8. $\left|-\frac{8}{12}\right| \quad \left|+\frac{2}{3}\right|$

9. $\left|-\frac{6}{10}\right| \quad +\frac{2}{5}$ 10. $|-3.1| \quad -6.2$

D. A student said, "You find the absolute value of a number by erasing the sign." What is wrong with this statement? Compare the statement above with:

> "See Patricia over there near the wall. She is too tall. To make her shorter all you have to do is write Pat."

E. In each of the following exercises two numerals are given. Copy each exercise and insert the symbols '=', '≠', '>',

We plan to include Review Exercises at the end of each unit. Review Exercises will include various kinds of exercises. Some of our purposes for the Review Exercises are:

1. Review of the content of the unit--exercises which include vocabulary and concepts.
2. Preparation for future units.
3. Further applications of the so-called "social", or "real life", or "practical" variety.
4. Exercises submitted by participating teachers and their students.

   (Note: We welcome exercises and other "pet" ideas you have used with your classes. We want you to encourage your students to submit problems to us. We'll incorporate their contributions in Review Exercises or in the text and attribute the contributions to them. Incidentally, if any of your students should indicate a desire to suggest an idea to us or to communicate in any way, please feel free to encourage them.)
5. Miscellaneous items of content which are considered part of high school mathematics largely for historical reasons, but which do not fit easily into the main developments of the unit itself.
6. Just about anything else we think of!

'<', '$\not>$', '$\not<$', '$\geqq$', and '$\leqq$', to form as many true statements as possible.

Sample. $|-6| \quad -3$

Solution. $|-6| \neq -3$

$|-6| > -3$

$|-6| \geqq -3$

$|-6| \not< -3$

1. $|-3| \quad |-4|$
2. $+5 \quad |-6|$
3. $|-4\frac{1}{2}| \quad +4\frac{1}{2}$
4. $|-3\frac{1}{5}| \quad +\frac{14}{5}$
5. $+1,506 \quad |+\frac{30010}{19}|$
6. $|-5| \quad 0$

REVIEW EXERCISES

In this set of exercises you will find problems which help you review what you have learned in this Unit. Also, you will find exercises reviewing your general knowledge of mathematics. In addition, some exercises will teach you something you did not previously know. For each exercise you should ask yourself, "Should I have learned this in the unit, did I already know it, or am I learning something new?"

A. Simplify each of the following expressions.

1. $(+3) + (-2)$
2. $(-3) - (+6)$
3. $(-3) - (-8)$
4. $(+6) + (-1)$
5. $(-12) + (-8)$
6. $(-16) - (+5)$
7. $(+27) - (-6)$
8. $5 - (-3)$
9. $(-5) + 7$
10. $(-15) + 3$
11. $(+1.3) - (+1.6)$
12. $(-5) + 3$
13. $(5.6) + (-2.3)$
14. $(-7.8) + (+2.4)$
15. $10.7 - (-3.4)$
16. $(-3.5) + (-2.9)$
17. $(+5) - (-3) + (-7)$
18. $(-2) - (-4) - (+3)$
19. $7 + 9 - (-2)$
20. $24 + (-6) - (-1)$
21. $(-13) + (+8) - (-9)$
22. $1.2 + (-1.7) + (-7.8)$
23. $(.5) \times 10$
24. $(-3) \times (-12)$
25. $(-7) \times (+9)$
26. $11 \times (-6)$

(continued on next page)

Some students had difficulty with exercises in Part B in which subtraction was involved. They had the notion that if the absolute value of a number were to be subtracted, the opposite of the absolute value could not be used. Discuss exercises such as 4 and 8 with the class, to avoid this kind of difficulty.

27. 6×12
28. $(-2) \times (-3) \times (+7)$
29. $11 \times (-3) \times (+4)$
30. $0 \times (-3) \times (-8)$
31. $(-12) \div (-3)$
32. $18 \div (-2)$
33. $(-24) \div (+8)$
34. $(-150) \div 25$
35. $0 \div (-5)$
36. $0 \div (+250)$
37. $(-27) \div 3$
38. $(.32) \div (-8)$
39. $(+48) \div (-12)$
40. $198 \div (-3)$

B. Tell which of the following statements are true and which are false.

1. $|-4| + |+4| = 0$
2. $|14| + |-4| = |-8|$
3. $|+6| + |-6| < |+6| + |+6|$
4. $|-3| - |+3| = 0$
5. $|-2| - (-3) = +5$
6. $|+5| + |+5| > |+5| + |-5|$
7. $(-4) + (-2) > -7$
8. $|-1| + (-3) - |-4| = -8$
9. $(+3) + |-2| - |-1| = +4$
10. $(-7) + |-2| + 0 = -50$
11. $(-7) + |-3| = -10$
12. $|-5| \geq |-6|$
13. $+10 = (-11) + 1$
14. $3 - 2 < 2 - 3$
15. $|+3| - |+2| = (+3) - (+2)$
16. $|+4| - |-2| < (+4) - (-2)$
17. $|+1| - |-2| \leq (+1) - (-2)$
18. $|-2| \times |-3| < (-2) \times (-3)$
19. $(-2) \times (+3) \not< (-2) \times (+3)$
20. $|+2| \times |+3| = (+2) \times (+3)$
21. $|+4 - (+7)| = |+7 - (+4)|$
22. $|+2 - (+13)| = |+13 - (+2)|$
23. $|-5 - (-3)| = |-3 - (-5)|$
24. $|-15 - (+2)| = |+2 - (-15)|$

C. Solve the following problems.

1. If a boy owes \$2.00, you may say that his financial status is -\$2.00. What is his financial status if he owes \$10.50? If he has no debt, and has \$8.75 on hand? If he has no debts and no money on hand?

2. In a game, a player has no points, and then loses 10 points. You may say that his standing is -10 points.

If a player has 5 points and loses 10 points, what is his point standing?

3. If the thermometer shows a temperature of +15° F., and then the temperature drops 27° F., what temperature does the thermometer then show?

4. If an elevator moves from the main floor three floors down, it is on the floor corresponding to -3. Assuming that the floor above the main floor is +1, what directed number corresponds to the floor at which the elevator stops after moving 11 floors down from the 7th floor?

5. The highest temperature on January 3 in Chicago was +10° F.; on January 4 it was 2° F. higher than on January 3; and on January 5, 5° F. lower than on January 4. What was the highest temperature on January 5?

6. A man gained $3.50 on one transaction and lost $17.25 on another. What was the net result of the two transactions? Use a directed number to indicate this result.

7. The temperature at 7 P.M. on a certain day in New York City was +7° F. At 3 A.M. of the next day, the temperature dropped to -9° F. State the change as a single directed number.

8. At the end of one year, a firm had a balance of -$10,700. At the end of the next year, the balance was +$15,400. How much better off was the firm at the end of the second year?

9. Mr. Jones bought five bonds at a par value of $250 each. One year later these bonds were listed at $237.50 each. Express the change in value for all five bonds as a directed number.

(continued on next page)

Part D contains a point which proved troublesome to some of our participants. We regard the symbol '63%', for example, as another name for $\frac{63}{100}$. Students in conventional courses come to feel that 63% is something slightly different than .63 or $\frac{63}{100}$, that you can't use 63% without thinking at the same time: "63% of something". However, in conventional texts you will also find tables with entries such as

$$25\% = \frac{1}{4}$$

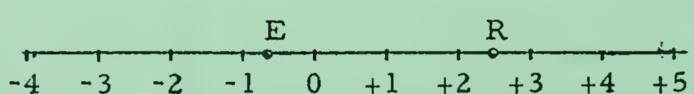
$$50\% = \frac{1}{2}$$

$$\vdots$$

To be consistent with our use of '=', we must regard the statement '$25\% = \frac{1}{4}$' as one which tells us that '25%' and '$\frac{1}{4}$' are names for the same number. Students need to be able to interpret '$7 + 25\% = 7\frac{1}{4}$' from our point of view, and to interpret 'the price of pencils has increased by 25%' from the point of view of everyday usage.

* * *

In doing the exercises in Part D, be sure the students do not think of the numbers 50%, $-\frac{7}{3}$, etc. as "names of points". The numbers are coordinates of points, which points we can "name" anything we like. For example:



'R' is the name of a point whose coordinate is 250%,

'E' is the name of a point on this number line; its coordinate is $-\frac{2}{3}$.

10. A certain number is added to -5 and the result is 16. What is the number?

11. The number -5 is subtracted from a certain number, and the result is +15. What is the number?

12. A certain number is multiplied by -4 and the result is -18. What is the number?

13. A certain number is divided by +6 and the result is -18. What is the number?

14. John owes Bill \$4.85. How much does John have to pay Bill on his debt in order to owe Bill only \$2.70?

D. You are probably accustomed to the use of percents only when other numbers are involved. Thus, you would say, "25% of 16". But what does the 'of' mean in this expression? It means the same as 'multiplied by'. When you say "$\frac{1}{2}$ of 50" you mean the same as "$\frac{1}{2} \times 50$". Now if the 'of' in '25% of 16' is replaced by '×' we obtain '25% × 16'. But when we multiply we must have two numbers and '25%' must be just another way of naming a number. Throughout this course we shall consider an expression such as '25%' to be another name for the number named by '$\frac{1}{4}$', '0.25', '$\frac{25}{100}$', etc.

Sketch a number line and mark the approximate location of points whose coordinates are listed.

1. $\frac{1}{2}$	2. $-\frac{1}{2}$	3. 50%	4. $+\frac{4}{3}$
5. $-\frac{7}{3}$	6. $+5\frac{1}{4}$	7. 30%	8. $-4\frac{3}{4}$
9. +0.6	10. +4.7	11.. 470%	12. -4.1
13. +3	14. $-133\frac{1}{3}\%$	15. -250%	16. +575%
17. +25%	18. -125%	19. +550%	20. 0%

Part E is another step in the campaign to get students to understand that a number can have many names. We have more exercises like this in subsequent units. This kind of exercise provides an interesting way to give drill. Both you and students can supplement this array with many more numerals.

E. Written below are many names for numbers. Pick one of the names and write it on a sheet of paper. Underneath it in a column write all of the other names given in the exercise that name the same number. Pick another name and repeat. Continue until you have used all of the names given.

$\frac{1}{2}$ $\quad$ $-\frac{6}{2}$ $\quad$ $\frac{31}{62}$ $\quad$ $-\frac{-8}{-16}$ $\quad$ 1

300% $\quad$ $+\frac{11}{3}$ $\quad$ $\frac{12}{12}$ $\quad$ 0.5 $\quad$ $\frac{22}{6}$

$\frac{-.0042}{-.0042}$ $\quad$ 50% $\quad$ $\frac{5}{11} \times \frac{121}{15}$ $\quad$ -3

$\frac{40 - (-4)}{12}$ $\quad$ $-\frac{-2}{4}$ $\quad$ $-1 - (-4)$ $\quad$ 100%

$\frac{-4}{-8}$ $\quad$ $\frac{9}{3} \times (-1)$ $\quad$ $+3\frac{2}{3}$ $\quad$ $\frac{1}{3} + \frac{1}{6}$

$-2 \times (-\frac{1}{2})$ $\quad$ $\frac{3 + 7}{6 + 4}$ $\quad$ $48\% + 2\%$ $\quad$ $\frac{16}{32} + 0$

$366\frac{1}{3}\%$ $\quad$ $4,832 - 4,831$

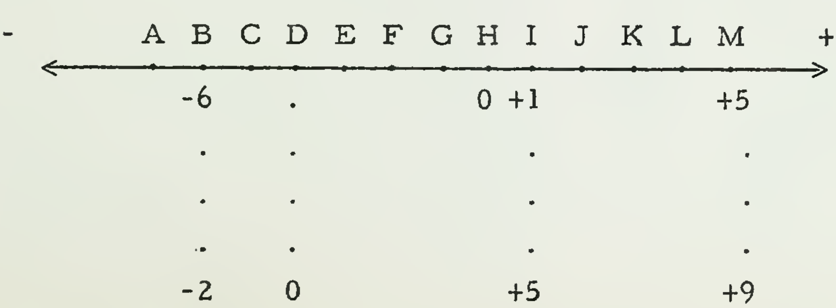
F. A number line has three important characteristics:

(a) a direction and its opposite;

(b) a unit length;

(c) an origin.

Any point on the number line may be used as an origin. Suppose the point H is chosen as the origin. Then the



Some students had difficulty with a type of exercise in which the points on a number line were named with letters <u>not</u> in alphabetical order. For this reason, we have prepared some supplementary exercises of that type. You may want to use these exercises after you have worked Part F.

<u>Supplementary exercises:</u>

Draw the number lines for the following trips.

a) A trip from point M to point N corresponds to +2, from N to P corresponds to -6, from P to D corresponds to -2.. If the coordinate for the point N is +5, what is the coordinate for the point D?

b) A trip from W to Q corresponds to -9, from Q to A corresponds to +7. If the coordinate for A is 4, what are the coordinates for the other points mentioned?

c) A trip from C to R corresponds to 14, from R to S corresponds to -32, from S to A corresponds to +10, from A to C corresponds to -8. Give the coordinates of the points A, C, S, if the coordinate of R is 3; -3; 0; $+\frac{1}{2}$; $-\frac{5}{6}$; -105.2; +3986.7.

d) Suppose that on a number line a trip from point A to point Z corresponds to -5, and a trip from point Z to point C corresponds to +9. If the coordinate of point A is -6, what is the coordinate of point C?

Mr. Fildes experienced difficulty with his students in working out the exercises in Part F. He suggests the table be rearranged in this fashion:

	Origin at	Change in Origin from F	Coordinates of Points:												
			H	I	E	A	F	L	C	B	K	M	D	J	G
a)	F	0	+2												
b)	B	-4													
c)						-9									
d)	A														
e)												0			

Let us know whether you think this arrangement contributes more to the students' understanding of the idea involved here.

coordinate of the point I, for example, is +1, of the point B is -6, of the point M is +5. Now, suppose the point D is chosen as the origin. In this case, the coordinate of the point I is +5, of the point B is -2, and of the point M is +9.

1. Complete the following table.

Origin at		Coordinates of points							
(a)	F	H: +2	I:	E:	A:	F:	L:	C:	B:
(b)	B	H:	I:	E:	A:	F:	L:	C:	B:
(c)		A: -9	F:	M:	K:	J:	I:	E:	C:
(d)	A	C:	K:	J:	I:	D:	G:	H:	L:
(e)		K:	J:	M:0	E:	A:	I:	D:	G:

2. Examine your answers in parts (a) and (b) of the preceding problem. In shifting the origin from point F to point B you changed the location of the origin by moving it 4 units in the negative direction; that is, we made a change of -4 units. Note the change in coordinates of the points H, I, E, A, etc. State a rule for finding new coordinates when the location of the origin has been changed by a given amount in a given direction. Check your rule by observing changes in the coordinate of the point I as the location of the origin changes in Exercise 1.

G. On the next page is a graph giving the mean (average) temperatures for each month of a certain year in Hawaii. Using the graph complete the following table and then answer questions 1 through 6.

Month	Mean Temperature	Month	Mean Temperature	Month	Mean Temperature
Jan.	68.9°	May		Sept.	
Feb.		June		Oct.	
March		July		Nov.	
April		Aug.		Dec.	

(2) Give five pairs of directed numbers such that the average of each pair is +6.

(3) Fill in the blank so that the average of each set is -2.

a) -6, -7, +12, ________

b) -9, +9, ________

c) $\frac{1}{2}$, $\frac{-1}{3}$, $\frac{-3}{4}$, ________

d) -3.0, -3.4, +2, +6.4, ________

e) +2, -2, -1, 0, -2, -2, +3, +1, ________

The answer to both Exercises 4 and 5 of Part G is 'yes'. To ensure that a student arrives at this answer (aside from arithmetic errors), it is important that he decide once and for all on the twelve numbers he "reads" as monthly temperatures from the graph. [Students may differ among each other in their readings.] He should then work with these twelve numbers for Exercises 1, 2, and 3, and not go back and "re-read" the graph.

Exercises 4 and 5 really have nothing to do with the given graph. They are concerned with averages of numbers and directed differences of those numbers from their average. [For example, students could read the graph entirely wrong and their answer to 4 would still be 'yes' if they computed correctly.]

Some students have difficulty in finding the average of a set of directed numbers. Here are some supplementary exercises which you might like to use after work on Part G has been completed.

(1) Find the average of each set of numbers.

a) 3.5, 6.09, 2.37, 100.12, 9.75

b) 3.89%, 12.1%, 7.14%, 8.0%

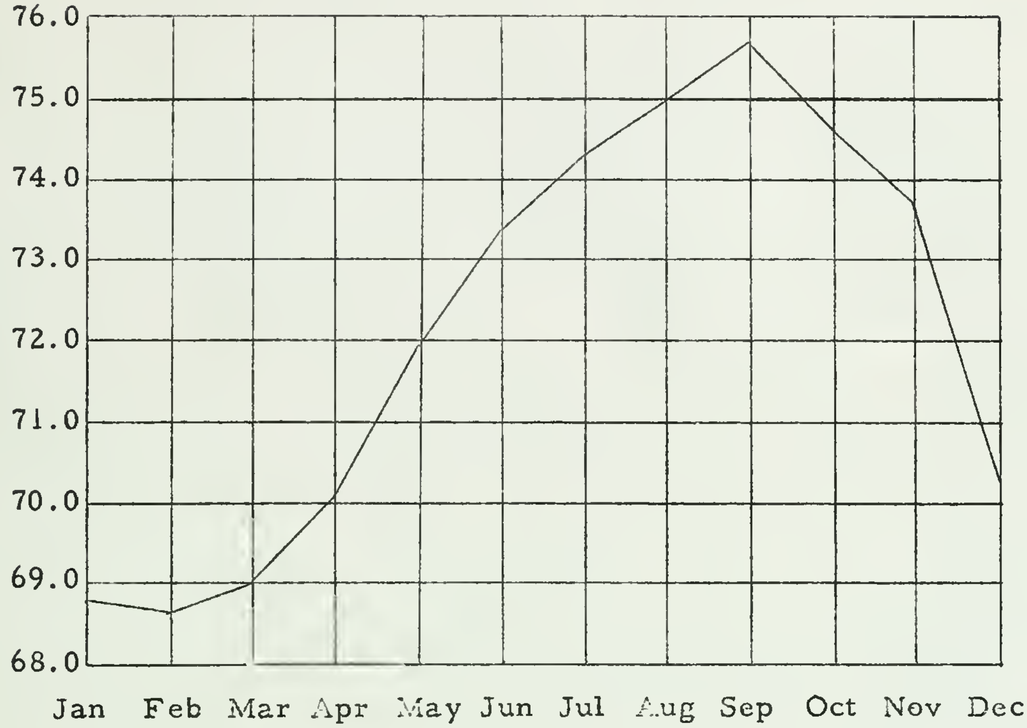
c) +5, -14

d) +3, -101, +105, -3

e) $\frac{-3}{8}$, $+\frac{1}{2}$, $\frac{-3}{4}$, $\frac{-15}{24}$

(continued on T. C. 73B)

Monthly Mean Temperatures in Hawaii



1. Find the mean of the mean monthly temperatures for the year.

2. Subtract the yearly mean temperature from the mean temperature for each month.

3. Add all the differences obtained in problem 2.

4. Is the result you obtained in problem 3 equal to 0 ?

5. Do you think that the sum of the differences between the mean of any set of numbers and each of the numbers is zero?

6. Is the average of the mean monthly temperatures for the year necessarily exactly the average temperature for the year? (Hint: do all the months have equal length?)

H. The bar graph on the next page shows mean January temperatures in Fahrenheit degrees computed over a period of years at some of the Alaska weather stations.

1. Give the mean of the January temperatures at the following weather stations: Ketchikan, Valdez, Susitna, Ruby, Tanana, and Dawson.

The last question in Exercise 7 is somewhat of a joke. As you know, we do not think that adolescents are necessarily concerned with the same kind of practical problem with which adults are concerned. We do not have any one answer in mind. Let your students exercise their imaginations. A few sample answers might be:

> A fuel company could estimate the amount of fuel that would be required, on the average, for a particular kind of house in this region and therefore could more nearly estimate the total amount of fuel they would be called upon to supply.
>
> A thermometer-making company, knowing that it is not economical to make thermometers which are accurate throughout their range might be able to guess the general region in which thermometers supplied to Alaska should be accurate.

There will be objections to these examples and others that may be suggested. Encourage debate and do not try to arrive at any definitive result. Probably the most important outcome of Exercise 7 is the realization that knowing just the mean temperature is not knowing very much at all.

2. According to the graph, how many degrees colder was Anchorage than Seward?

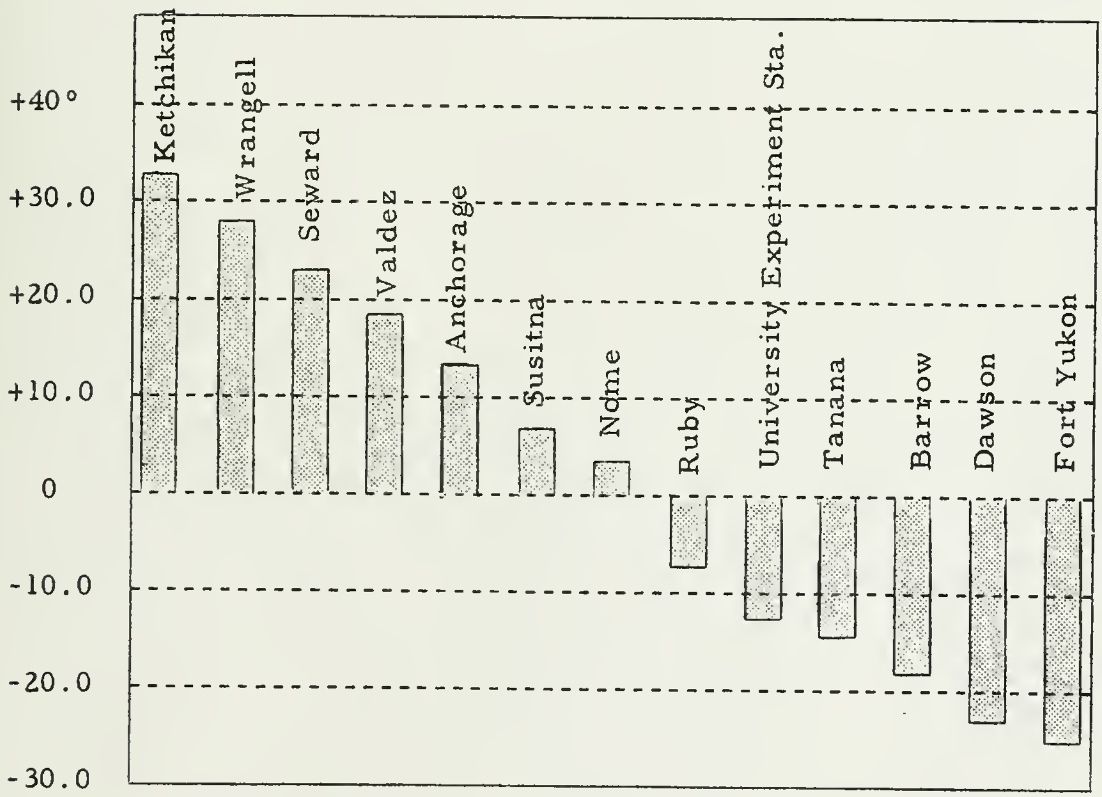
3. What station had the temperature closest to 0? What was this temperature?

4. How many degrees warmer was Ruby than Barrow?

5. What was the warmest station?

6. How many stations had their mean January temperature above the freezing point? How many below the freezing point?

7. The mean temperature for the thirteen stations is +2.8°
What practical use could be made of this fact?



Bar Graph Showing Fahrenheit Temperatures
at Various Alaskan Weather Stations

I. Punctuate the following paragraph with single quotes:

The boy was making a list of the class. First he wrote Jim. Beside Jim he wrote 1. Then he wrote Helen and beside Helen he wrote 3. He looked at the 3 and realized that he had made an error. He erased the 3 and this time put 2 beside Helen. He continued. After writing 13 names he wrote 14. When he finished the list he had 23 names and 23 numerals. He thought, "There are 13 boys and 10 girls, and, since 10 + 13 = 23, I have finished the list."